

Some results on the Signature and Cubature of the Fractional Brownian motion for $H > \frac{1}{2}$

Riccardo Passeggeri*

Abstract

In this work we present four different results concerning the signature and the cubature of fractional Brownian motion (fBm). The first result regards the rate of convergence of the dyadic approximation of the expected signature of the fBm to its exact value, for a value of the Hurst parameter $H \in (\frac{1}{2}, 1]$. We show that the rate of convergence is given by 2^{-2Hm} , where 2^{-m} is the size of a single time step of the dyadic approximation. We believe that this rate is sharp as it is consistent with the result of Ni and Xu [15], who showed that the sharp rate of convergence for the Brownian motion (i.e. fBm with $H = \frac{1}{2}$) is given by 2^{-m} . The second result regards the bound of the *coefficient* of the rate of convergence obtained in the first result. For this result we also analyse the behaviour of this bound as the number of iterated integrals of the signature of fBm goes to infinity, showing that the bound of the $2k$ -th term in the signature is uniformly bounded by $\frac{AH(2H-1)}{(k-1)!2^k}$, where A is a constant and k is the number of iterated integrals. The third result regards the bound of the expected signature of the fBm. We show that the bound for the $2k$ -th term of the signature is simply given by $\frac{1}{k!2^k}$. This is a sharper estimate than the one obtained by Chevyrev and Lyons in [5], who showed that the expected signature has infinite radius of convergence, but not a factorial decay, and than the one by Friz and Riedel [9], who proved a factorial decay of $\frac{1}{(k/2)!}$. The last result concerns the cubature formula for the one dimensional fBm for $H > \frac{1}{2}$ up to degree 5. This result extends the work of Lyons and Victoir [13] who focused on the multidimensional Brownian motion cubature formula.

Key words: fractional Brownian motion, signature, rate of convergence, cubature method.

1 Introduction

The signature of a d -dimensional fractional Brownian motion is a sequence of iterated Stratonovich integrals along the paths of the fractional Brownian motion; it is an object taking values in the tensor algebra over \mathbb{R}^d . Signatures were firstly studied by K.T.-Chen in 1950s in a series of papers [2], [3] and [4]. In the last twenty years the attention devoted to signatures has increased rapidly. This has been caused by the pivotal role they have in rough path theory, a field developed in the late nineties by Terry Lyons culminating in the paper [11], which is also at the base of newly developed theory of regularity structures [10]. The signature of a path summarises the essential properties of that path allowing the possibility to study SPDEs driven by that path.

In 2015 Hao Ni and Weijun Xu [15] computed the sharp rate of convergence for expected Brownian signatures. They obtained a rate of convergence of 2^{-m} , where 2^{-m} is the size of the mesh of the dyadic approximation.

However, for fractional Brownian motion no progress has been made in this direction. In particular, the rate of convergence for the expected signature of the fractional Brownian motion is not known for any value of the Hurst parameter $H \in [0, 1]$. This article address this problem, obtaining the rate of convergence for $H \in (\frac{1}{2}, 1]$. In order to obtain it we used the results of Baudoin and Coutin [1]. Indeed, in 2007 Baudoin and

*Imperial College London, UK, and University of Reading, UK. Email: riccardo.passeggeri14@imperial.ac.uk

Coutin computed the expected signature for fractional Brownian motion for $H > \frac{1}{2}$ and also for small times for $H \in (\frac{1}{3}, \frac{1}{2})$. Further works that analyse the properties of the signature of the fBm are [6] and [14] among others.

In this work we focus on the weak rate of convergence and we refer to the work of Friz and Riedel [8] for the strong rate of convergence. They obtained a rate of 2^{-Hm} , while here we obtain a weak convergence rate of 2^{-2Hm} .

Moving away from the dyadic approximation and focusing just on the expected signature, we recall the work of Chevyrev and Lyons [5]. In [5] they showed that the expected signature has infinite radius of convergence, but not a factorial decay. In this work we show that the expected signature has a factorial decay, indeed the bound for the $2k$ -th term of the signature is simply given by $\frac{1}{k!2^k}$ for all H . The sharp decay rate is expected to be $\frac{1}{(2kH)!}$, but it remains an open problem. In the $H > \frac{1}{2}$ case, our result gives an alternative proof, with sharper estimates, that the expected signature of fractional Brownian motion has infinite radius of convergence, which by [5] implies that the expected signature determines the signature in distribution. Our estimate is also sharper than the one obtained by Friz and Riedel [9], who proved a factorial decay of $\frac{1}{(k/2)!}$.

In 2003 Lyons and Victoir [13] developed a numerical methods for solving parabolic PDEs and general SDEs driven by Brownian motions called cubature method on Wiener space. In the cubature method the first step is to obtain the cubature formula in which the truncated signature of a path (in the case of [13] is the Brownian motion) is matched to a finite sum of Dirac delta measures applied to the iterated integrals of deterministic paths. In this work we give an extension of this result by obtaining the cubature formula for the fractional Brownian motion for a particular case.

This paper is structured in the following way. In section 2 we introduce some notations and state the main results. In section 3 we will discuss about preliminaries involving definitions and the results of other authors. In section 4, 5 and 6 we prove the first three main results of the article. In section 7 we discuss the fourth result whose proof is in the Appendix due to its long computations.

2 Main Results

In this section we introduce the main results of the article. But first, we introduce some notations. The notations used in this work are in line with the ones used in the papers of Baudoin and Coutin [1], Lyons [11], and Lyons and Victoir [13] and in the book by Lyons, Caruana, and Lévy [12].

The Fractional Brownian motion is defined as follows.

Definition 2.1. *Let H be a constant belonging to $(0,1)$. A fractional Brownian motion (fBm) $(B^H(t))_{t \geq 0}$ of Hurst index H is a continuous and centered Gaussian process with covariance function*

$$\mathbb{E} [B^H(t)B^H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$$

From now on we will denote $B^H := B$.

For $H = \frac{1}{2}$ then the fBm is a Bm. Further, the multi-dimensional fractional Brownian motion has coordinate components that are independent and identically distributed copies of one dimensional fBm.

Now, we define the simplex $\Delta^k[0, 1]$,

$$\Delta^k[0, 1] := \{(t_1, \dots, t_k) \in [0, 1]^k : t_1 < \dots < t_k\}.$$

Further, we define the following iterated integrals. Let $I = (i_1, \dots, i_k) \in \{1, \dots, d\}^k$ be a word with length k then

$$\int_{\Delta^k[0, 1]} dB^I := \int_{0 \leq t_1 < \dots < t_k \leq 1} dB_{t_1}^{i_1} \dots dB_{t_k}^{i_k}$$

and

$$\int_{\Delta^k[0, 1]} dB^{m, I} := \int_{0 \leq t_1 < \dots < t_k \leq 1} dB_{t_1}^{m, i_1} \dots dB_{t_k}^{m, i_k}$$

where B is the fractional Brownian motion with Hurst parameter H and B^m is its dyadic approximation. In addition, B^i is the i -th coordinate component of the fBm B and the iterated integrals can be defined in the sense of Young [16].

Moreover, the dyadic approximation B^m is defined as follows. Let $t_i^m = i2^{-m}$ for $i = 0, \dots, 2^m$. If $t \in [t_i^m, t_{i+1}^m]$ then

$$B_t^m = B_{t_i^m}^m + 2^m(t - t_i^m)(B_{t_{i+1}^m}^m - B_{t_i^m}^m)$$

From now on we will denote $t_i^m := t_i$.

We can now present our main results. The first result is about the rate of convergence of the expected signature of the dyadic approximation of the fBm to its exact value.

Theorem 2.2. *Let $H > \frac{1}{2}$. Letting $I = (i_1, \dots, i_{2k})$ be a word where $i_l \in \{1, \dots, d\}$ for $l = 1, \dots, 2k$, then for all m*

$$\left| \mathbb{E} \left(\int_{\Delta^{2k}[0, 1]} dB^I \right) - \mathbb{E} \left(\int_{\Delta^{2k}[0, 1]} dB^{m, I} \right) \right| \leq C 2^{-2Hm}$$

where C is a finite constant and independent of m .

It is important to stress that for the Brownian motion case, i.e. $H = \frac{1}{2}$, Ni and Xu in [15] proved that the sharp rate of convergence is given by 2^{-m} , which is in line with the result presented here. Moreover, the proof used in [15] cannot in principle be used to prove our result since Ni and Xu used the independence of the increments property (i.e. the semigroup property) of the Brownian motion, which does not hold for the fBm. Conversely, the proof used to prove **Theorem 2.2** is based on the integral form of the covariance function of the fBm which is valid only for $H > \frac{1}{2}$, hence our proof cannot in principle be used to prove the result in [15].

In the next theorem we focus on and refine the value of the coefficient C of the previous theorem and we provide a bound for it.

Theorem 2.3. *Let $H > \frac{1}{2}$. Letting $I = (i_1, \dots, i_{2k})$ be a word where $i_l \in \{1, \dots, d\}$ for $l = 1, \dots, 2k$, then*

$$\limsup_{m \rightarrow \infty} 2^{2Hm} \left| \mathbb{E} \left(\int_{\Delta^{2k}[0, 1]} dB^I \right) - \mathbb{E} \left(\int_{\Delta^{2k}[0, 1]} dB^{m, I} \right) \right| \leq \frac{AH(2H-1)}{(k-1)!2^k}$$

where A is a finite constant and independent of m and k .

The following theorem provides a bound for the value for the expected signature of the fBm. As mentioned in the introduction the following result is a sharper estimates than the one obtained by Chevyrev and Lyons [5] and the one by Friz and Riedel [9]. Our result in particular implies, in the $H > \frac{1}{2}$ case, Chevyrev-Lyons' result that the expected signature of fBm has infinite radius of convergence. This in turns implies that the expected signature determines the signature in distribution.

Theorem 2.4. *Let $H > \frac{1}{2}$. Letting $I = (i_1, \dots, i_{2k})$ be a word where $i_l \in \{1, \dots, d\}$ for $l = 1, \dots, 2k$, then*

$$\mathbb{E} \left(\int_{\Delta^{2k}[0,1]} dB^I \right) \leq \frac{1}{k!2^k}$$

We move now to the cubature method. We provide the cubature formula for the one dimensional fBm up to certain degrees. Moreover, we will discuss in more details the concept of *degree* of a cubature formula in chapter 7.

Theorem 2.5. *Let $H \geq \frac{1}{2}$. Define \hat{B} and $\hat{\omega}$ to be*

$$\hat{B}_{t_l}^{i_l} = \begin{cases} t_l, & \text{if } i_l = 0, \\ B_{t_l}, & \text{if } i_l = 1. \end{cases} \quad \text{and} \quad \hat{\omega}_{t_l, j}^{i_l} = \begin{cases} t_l, & \text{if } i_l = 0, \\ \omega_{t_l, j}, & \text{if } i_l = 1. \end{cases}$$

for $l = 1, \dots, k$. The weights $\{\lambda_1, \lambda_2, \lambda_3\}$ and the paths $\{\omega_{t,1}, \omega_{t,2}, \omega_{t,3}\}$ will satisfy the cubature formula

$$\mathbb{E} \left[\int_{0 < t_1 < \dots < t_k < T} d\hat{B}_{t_1}^{i_1} \cdots d\hat{B}_{t_k}^{i_k} \right] = \sum_{j=1}^3 \lambda_j \int_{0 < t_1 < \dots < t_k < T} d\hat{\omega}_{t_1, j}^{i_1} \cdots d\hat{\omega}_{t_k, j}^{i_k},$$

for the following degree

$$\text{Degree} = \begin{cases} 5 & \text{for } \frac{1}{2} \leq H < \frac{2}{3}, \\ 4 & \text{for } \frac{2}{3} \leq H < 1, \\ 3 & \text{for } H = 1, \end{cases}$$

if $\lambda_1 = \lambda_2 = \frac{1}{6}$ and $\lambda_3 = \frac{2}{3}$, and

$$\omega_{t,1} = \begin{cases} (2\alpha - \beta)t, & t \in [0, \frac{1}{3}], \\ (\alpha - \beta) + (2\beta - \alpha)t, & t \in [\frac{1}{3}, \frac{2}{3}], \\ (\beta - \alpha) + (2\alpha - \beta)t, & t \in [\frac{2}{3}, 1], \end{cases}$$

where

$$\alpha := \frac{2H\sqrt{3} + \sqrt{3}}{2H + 1} \quad \text{and} \quad \beta := \frac{\sqrt{-96H^2 + 66H + 57}}{2H + 1},$$

and $\omega_{t,2} = -\omega_{t,1}$ and $\omega_{t,3} = 0$ for $t \in [0, 1]$.

It is important to remark that for $H = \frac{1}{2}$, i.e. when the fractional Brownian motion is just a Brownian motion, the results obtained in this theorem perfectly coincide with the results of Lyons and Victoir obtained in [13].

3 Preliminaries

One of the main concepts of this paper is the signature of the fractional Brownian motion. In order to give a definition of signature we need to introduce first the following concept: the p -variation of a path.

Definition 3.1. Let $p \geq 1$ be a real number. Let $X : J \rightarrow E$ be continuous path, where J is a compact interval and E is a Banach space. The p -variation of X on the interval J is defined by

$$\|X\|_{p,J} = \left[\sup_{\mathcal{D} \subset J} \sum_{i=0}^{r-1} |X_{t_i} - X_{t_{i+1}}|^p \right]^{\frac{1}{p}}$$

where the supremum is taken over all the partitions of J .

A path X is said to be of finite p -variation over the interval J if $\|X\|_{p,J} < \infty$.

For signature of a stochastic process we mean the following

Definition 3.2. Let E be a Banach space. Let $X : [0, T] \rightarrow E$ be a continuous path with finite p -variation for some $p < 2$. The signature of X is the element $S(X)$ of $T(E) := \{\mathbf{a} = (a_0, a_1, \dots) | \forall n \geq 0, a_n \in E^{\otimes n}\}$ defined as follows

$$S(X_{[0,T]}) = (1, X_{[0,T]}^1, X_{[0,T]}^2, \dots)$$

where, for each $n \geq 1$,

$$X_{[0,T]}^n = \int_{0 < u_1 < \dots < u_n < T} dX_{u_1} \otimes \dots \otimes dX_{u_n}.$$

The integration in the previous definition is defined in the sense of Young [16].

The fractional Brownian motion is a p -variation path for all $p > \frac{1}{H}$. Hence, for $H \in (\frac{1}{2}, 1]$ we have $p < 2$. Notice that for I , a word composed by the letters (i_1, \dots, i_{2k}) where each $i_l \in 1, \dots, d$, we have that $dB^I := dB^{i_1} \dots dB^{i_{2k}}$. Therefore, we have that the $2k$ -th element of the signature of the fBm and its dyadic approximation are respectively given by

$$\mathbb{E} \left(\int_{0 < u_1 < \dots < u_{2k} < 1} dB_{u_1} \otimes \dots \otimes dB_{u_{2k}} \right) = \sum_{i_1=1}^d \dots \sum_{i_{2k}=1}^d \mathbb{E} \left(\int_{\Delta^{2k}[0,1]} dB^I \right) e_{i_1} \otimes \dots \otimes e_{i_{2k}} \quad (1)$$

and by

$$\mathbb{E} \left(\int_{0 < u_1 < \dots < u_{2k} < 1} dB_{u_1}^m \otimes \dots \otimes dB_{u_{2k}}^m \right) = \sum_{i_1=1}^d \dots \sum_{i_{2k}=1}^d \mathbb{E} \left(\int_{\Delta^{2k}[0,1]} dB^{m,I} \right) e_{i_1} \otimes \dots \otimes e_{i_{2k}} \quad (2)$$

where (e_1, \dots, e_d) is the basis of \mathbb{R}^d .

We recall two results which can be found for example in Baudoin and Coutin paper [1]. First, we present a reformulation of the Isserlis' (or Wick's) theorem:

Lemma 3.3. Let $G = (G_1, \dots, G_{2k})$ be a centered Gaussian vector. We have

$$\mathbb{E}(G_1 \dots G_{2k}) = \frac{1}{k! 2^k} \sum_{\sigma \in \mathcal{G}_{2k}} \prod_{l=1}^k \mathbb{E}(G_{\sigma(2l-1)} G_{\sigma(2l)}) \quad (3)$$

where \mathcal{G}_{2k} is the group of the permutations of the set $\{1, \dots, 2k\}$.

Therefore from the previous lemma we have

$$\mathbb{E} \left(\int_{\Delta^{2k}[0,1]} dB^{m,I} \right) = \frac{1}{k!2^k} \sum_{\sigma \in \mathcal{G}_{2k}} \int_{\Delta^{2k}[0,1]} \prod_{l=1}^k \mathbb{E} \left(\frac{dB^{m,i_{\sigma(2l-1)}}}{dt_{\sigma(2l-1)}} \frac{dB^{m,i_{\sigma(2l)}}}{dt_{\sigma(2l)}} \right) dt_1 \cdots dt_{2k} \quad (4)$$

Notice that if $t_{\sigma(2l-1)} \in [t_j, t_{j+1}]$ and $t_{\sigma(2l)} \in [t_i, t_{i+1}]$ then

$$\mathbb{E} \left(\frac{dB^{m,i_{\sigma(2l-1)}}}{dt_{\sigma(2l-1)}} \frac{dB^{m,i_{\sigma(2l)}}}{dt_{\sigma(2l)}} \right) = \delta_{i_{\sigma(2l)}, i_{\sigma(2l-1)}} H(2H-1) 2^{2m} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dx dy \quad (5)$$

Further, recall Theorem 31 of [1].

Theorem 3.4. Assume $H > \frac{1}{2}$. Letting $I = (i_1, \dots, i_{2k})$ be a word, then

$$\mathbb{E} \left(\int_{\Delta^{2k}[0,1]} dB^I \right) = \frac{H^k (2H-1)^k}{k!2^k} \sum_{\sigma \in \mathcal{G}_{2k}} \int_{\Delta^{2k}[0,1]} \prod_{l=1}^k \delta_{i_{\sigma(2l)}, i_{\sigma(2l-1)}} |t_{\sigma(2l)} - t_{\sigma(2l-1)}|^{2H-2} dt_1 \cdots dt_{2k} \quad (6)$$

We now move toward the cubature method and we introduce two definitions of the cubature formula. Both are introduced in the work of Lyons and Victoir [13].

We denote by $\mathbb{R}_m[X_1, \dots, X_d]$ the space of polynomials in d variables and of degree less than m .

Definition 3.5. Let μ be a positive measure on \mathbb{R}^d , $d < \infty$ and m be a natural number. We say that the points x_1, \dots, x_n in the support of μ , and the positive weights $\lambda_1, \dots, \lambda_n$ define a cubature formula of degree m with respect to μ if and only if, for all polynomials $P \in \mathbb{R}_m[X_1, \dots, X_d]$,

$$\int_{\mathbb{R}^d} P(x) \mu(dx) = \sum_{i=1}^n \lambda_i P(x_i).$$

When $d = 1$, one talks about quadrature formulae rather than cubature formulae.

Now we move from a finite dimensional space to the Wiener space, which is infinite dimensional.

Definition 3.6. Let m be a natural number. Let define $\mathcal{A}_m := \{(i_1, \dots, i_k) \in \{0, \dots, d\}^k, k + \text{card}\{j, i_j = 0\} \leq m\}$. We say that the paths

$$\hat{\omega}_1, \dots, \hat{\omega}_n \in C_{0,bv}^0([0, T], \mathbb{R}^d)$$

and the positive weights $\lambda_1, \dots, \lambda_n$ define a cubature formula on Wiener space of degree m at time T , if and only if, for all $(i_1, \dots, i_k) \in \mathcal{A}_m$,

$$\mathbb{E} \left[\int_{0 < t_1 < \dots < t_k < T} d\hat{B}_{t_1}^{i_1} \cdots d\hat{B}_{t_k}^{i_k} \right] = \sum_{j=1}^n \lambda_j \int_{0 < t_1 < \dots < t_k < T} d\hat{\omega}_{j,t_1}^{i_1} \cdots d\hat{\omega}_{j,t_k}^{i_k}.$$

Notice that we wrote $\hat{B}_t^{i_l}$ instead of $B_t^{i_l}$ for $l = 1, \dots, k$ in order to stress the fact that it is not a Brownian motion, since it is a Brownian motion for $i_l \neq 0$ but for $i_l = 0$ we have $\hat{B}_t^0 = t$.

In general, we say that the paths $\hat{\omega}_1, \dots, \hat{\omega}_n$ and the weights $\lambda_1, \dots, \lambda_n$ define a cubature formula on Wiener

space of degree m if the expectations of the (Stratonovich)-iterated integrals of degree less than m are, under the Wiener measure and under the probability measure

$$\mathbb{Q} = \sum_{i=1}^n \lambda_i \delta_{\omega_i}$$

the same. In this article, we do not deal with the concatenation step of the cubature method since we focus mainly in obtaining the cubature formula for the fractional Brownian motion. Indeed, while the concatenation for the Brownian motion case takes advantage of the independence of the increments (and so of the Markov semigroup property), for the fractional Brownian motion case these properties do not hold. Hence, the problem of the concatenation is an open problem and appears to be not at all trivial.

Remark 3.7. *For the fractional Brownian motion we have the same definition of cubature formula as for the Brownian motion except for the definition of \mathcal{A}_m . Indeed, for the fBm we have*

$$\mathcal{A}_m := \{(i_1, \dots, i_k) \in \{0, \dots, d\}^k, k + (\frac{1}{H} - 1) \times \text{card}\{j, i_j = 0\} \leq m\}.$$

As it is possible to see the two definitions coincide for $H = \frac{1}{2}$.

4 Problem 1: Rate of convergence

The first result we show regards rate of convergence of the expected signature of the dyadic approximation of the fractional Brownian motion (fBm) to its exact value. The sharpness of the rate of convergence is a delicate matter and it will be discussed at the end of this chapter.

In this chapter we prove theorem 2.2. As mentioned before, this theorem covers the weak convergence rate of the signature of the fBm. A previous result on the strong convergence rate was obtained by Friz and Riedel in [8].

Recall from theorem 3.4 that we have

$$\begin{aligned} & \left| \mathbb{E} \left(\int_{\Delta^{2k}[0,1]} dB^I \right) - \mathbb{E} \left(\int_{\Delta^{2k}[0,1]} dB^{m,I} \right) \right| \\ &= \left| \frac{H^k (2H-1)^k}{k! 2^k} \sum_{\sigma \in \mathcal{G}_{2k}} \int_{\Delta^{2k}[0,1]} \prod_{l=1}^k \delta_{i_{\sigma(2l)}, i_{\sigma(2l-1)}} |t_{\sigma(2l)} - t_{\sigma(2l-1)}|^{2H-2} \right. \\ & \quad \left. - \prod_{l=1}^k \delta_{i_{\sigma(2l)}, i_{\sigma(2l-1)}} \sum_{i,j=1}^{2m} \mathbf{1}_{[t_i, t_{i+1}] \times [t_j, t_{j+1}]}(t_{\sigma(2l)}, t_{\sigma(2l-1)}) 2^{2m} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dx dy dt_1 \cdots dt_{2k} \right| \end{aligned}$$

In order to understand this as a whole we need to first understand the single parts of it. In other words let us focus on

$$\int_{u < s < t < v} \left(|t-s|^{2H-2} - \sum_{i,j=1}^{2m} \mathbf{1}_{[t_i, t_{i+1}] \times [t_j, t_{j+1}]}(t, s) 2^{2m} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dx dy \right) ds dt$$

Now we are going to split this double integral into different parts according to our dyadic approximation. In particular, let t_q be the lowest dyadic time above u and t_p the highest dyadic time below v , as shown in the Figure 1. Then we have that our double integral can be rewritten in the following way:

$$\int_{u < s < t < v} = \int_u^v \int_u^t = \int_u^{t_q} \int_u^t + \int_{t_q}^{t_p} \int_u^{t_q} + \int_{t_p}^v \int_u^{t_q} + \int_{t_p}^v \int_{t_q}^{t_p} + \int_{t_p}^v \int_{t_p}^t + \int_{t_q}^{t_p} \int_{t_q}^t \quad (7)$$

To better understand this decomposition look at the upper triangle in Figure 1 (i.e. the triangle from the red diagonal to the up and left blue edges). The first integral is the small triangle in the lower left. From here you first move up and then move right. The last integral is the central yellow triangle in Figure 1. For this integral the dyadic approximation and the exact value are equal, hence their difference is zero. This is because the dyadic approximation is equal to the exact value if the points in time considered coincide with the dyadic points.

What we want to show now is that for the other integrals the difference between the dyadic approximation and the exact value is going to zero as the dyadic time step goes to zero (i.e. $m \rightarrow \infty$).

In order to facilitate the reading and understanding of the content of this chapter, instead of having a 7 page long proof we decided to split the proof of theorem 2.2 in different small propositions.

The following propositions regards the rate of convergence of each double integral of the right hand side of equation (7).

So let us focus on the integrals: $\int_u^{t_q} \int_u^t$ and $\int_{t_p}^v \int_{t_p}^t$. We have the following proposition.

Proposition 4.1. *The differences*

$$\left| \int_u^{t_q} \int_u^t \left(|t-s|^{2H-2} - \sum_{i,j=1}^{2^m} \mathbf{1}_{[t_i, t_{i+1}] \times [t_j, t_{j+1}]}(t, s) 2^{2m} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dx dy \right) ds dt \right|$$

and

$$\left| \int_{t_p}^v \int_{t_p}^t \left(|t-s|^{2H-2} - \sum_{i,j=1}^{2^m} \mathbf{1}_{[t_i, t_{i+1}] \times [t_j, t_{j+1}]}(t, s) 2^{2m} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dx dy \right) ds dt \right|$$

are bounded by

$$\frac{2^{-2Hm}}{H(2H-1)}.$$

Proof.

$$\begin{aligned} & \int_u^{t_q} \int_u^t \left(|t-s|^{2H-2} - \sum_{i,j=1}^{2^m} \mathbf{1}_{[t_i, t_{i+1}] \times [t_j, t_{j+1}]}(t, s) 2^{2m} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dx dy \right) ds dt \\ &= \int_u^{t_q} \int_u^t \left(|t-s|^{2H-2} - 2^{2m} \int_{t_{q-1}}^{t_q} \int_{t_{q-1}}^{t_q} |x-y|^{2H-2} dx dy \right) ds dt \\ &= \frac{(t_q - u)^{2H}}{2H(2H-1)} - 2^{2m} \int_u^{t_q} \int_u^t \frac{(t_q - t_{q-1})^{2H}}{H(2H-1)} ds dt \end{aligned}$$

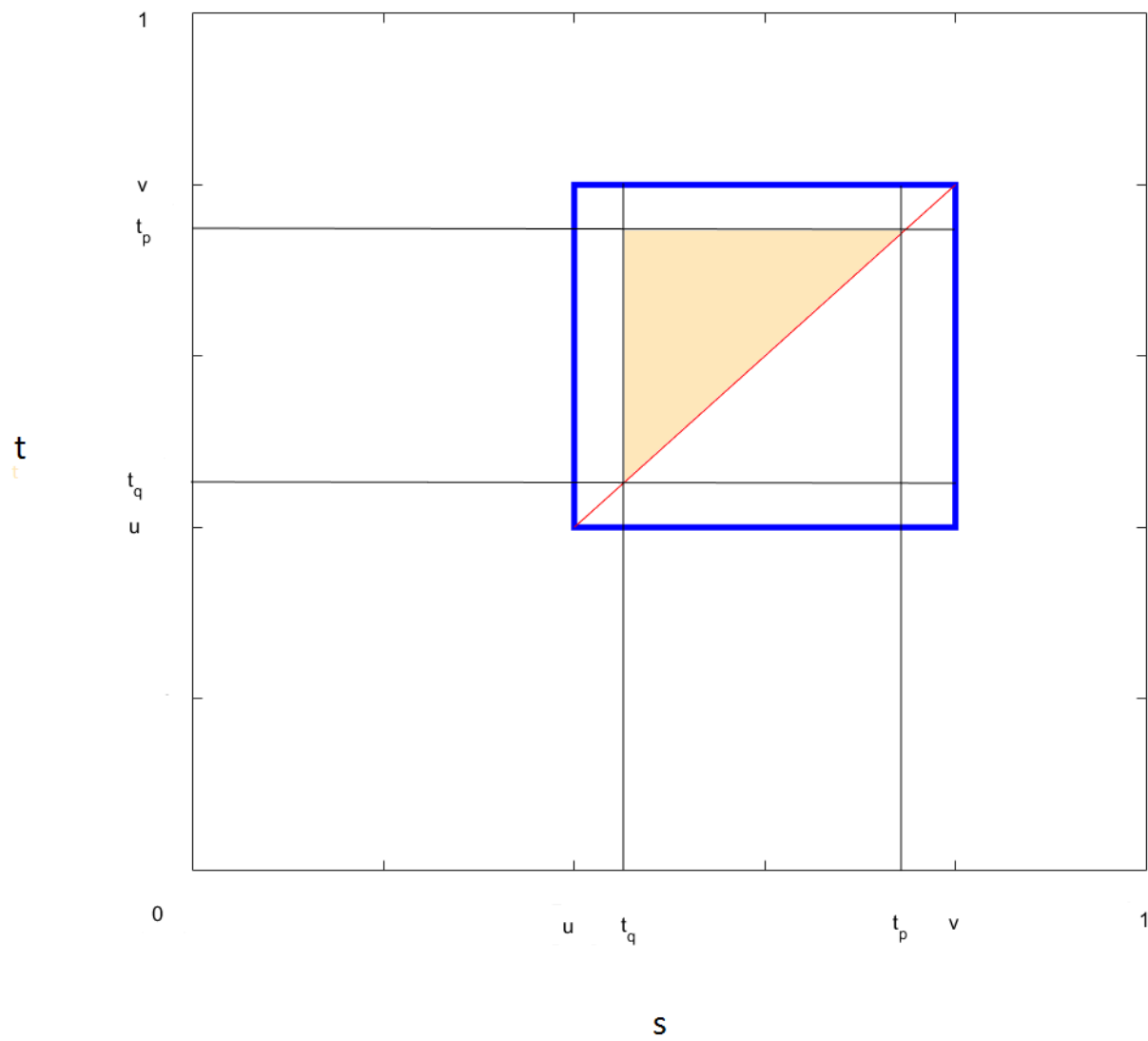


Figure 1: Representation of the area of the double integral (7).

$$= \frac{(t_q - u)^{2H}}{2H(2H-1)} - 2^{2m-2mH} \frac{(t_q - u)^2}{2H(2H-1)}$$

Notice that

$$\begin{aligned} \left| \frac{(t_q - u)^{2H}}{2H(2H-1)} - 2^{2m-2mH} \frac{(t_q - u)^2}{2H(2H-1)} \right| &\leq \frac{(t_q - u)^{2H}}{2H(2H-1)} + 2^{2m-2mH} \frac{(t_q - u)^2}{2H(2H-1)} \\ &\leq \frac{2^{-2Hm}}{2H(2H-1)} + 2^{2m-2mH} \frac{2^{-2m}}{2H(2H-1)} = \frac{2^{-2Hm}}{H(2H-1)} \end{aligned}$$

The same reasoning done for this integral applies *mutatis mutandis* to the integral $\int_{t_p}^v \int_{t_p}^t$. □

Let us focus on the integrals: $\int_{t_q}^{t_p} \int_u^{t_q}$ and $\int_{t_p}^v \int_{t_q}^{t_p}$. We have the following proposition

Proposition 4.2. *The differences*

$$\left| \int_{t_q}^{t_p} \int_u^{t_q} \left(|t-s|^{2H-2} - \sum_{i,j=1}^{2^m} \mathbf{1}_{[t_i, t_{i+1}] \times [t_j, t_{j+1}]}(t, s) 2^{2m} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dx dy \right) ds dt \right|$$

and

$$\left| \int_{t_p}^v \int_{t_q}^{t_p} \left(|t-s|^{2H-2} - \sum_{i,j=1}^{2^m} \mathbf{1}_{[t_i, t_{i+1}] \times [t_j, t_{j+1}]}(t, s) 2^{2m} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dx dy \right) ds dt \right|$$

are bounded by

$$\left(\frac{2^{2H} + 2}{H(2H-1)} + (4-4H) \sum_{i=1}^{\infty} (i)^{2H-3} \right) 2^{-2Hm}$$

for $H \in (\frac{1}{2}, 1)$, and by

$$\left(\frac{2^{2H} + 2}{H(2H-1)} \right) 2^{-2Hm}$$

for $H = 1$.

Proof. We have

$$\begin{aligned} &\int_{t_q}^{t_p} \int_u^{t_q} \left(|t-s|^{2H-2} - \sum_{i,j=1}^{2^m} \mathbf{1}_{[t_i, t_{i+1}] \times [t_j, t_{j+1}]}(t, s) 2^{2m} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dx dy \right) ds dt \\ &= \int_{t_q}^{t_p} \int_u^{t_q} \left(|t-s|^{2H-2} - \sum_{i=t_q}^{t_{p-1}} \mathbf{1}_{[t_i, t_{i+1}]}(t) 2^{2m} \int_{t_i}^{t_{i+1}} \int_{t_{q-1}}^{t_q} |x-y|^{2H-2} dx dy \right) ds dt \\ &= \sum_{i=t_q}^{t_{p-1}} \int_{t_i}^{t_{i+1}} \int_u^{t_q} \left(|t-s|^{2H-2} - 2^{2m} \int_{t_i}^{t_{i+1}} \int_{t_{q-1}}^{t_q} |x-y|^{2H-2} dx dy \right) ds dt \end{aligned}$$

Here we need to split the sum in two parts. The first part is composed by the first element of the sum, while the second is composed by the whole sum without the first element. Let us first consider the first part.

$$\int_{t_q}^{t_{q+1}} \int_u^{t_q} \left(|t-s|^{2H-2} - 2^{2m} \int_{t_q}^{t_{q+1}} \int_{t_{q-1}}^{t_q} |x-y|^{2H-2} dx dy \right) ds dt$$

$$\begin{aligned}
&= \frac{(t_{q+1} - u)^{2H} - (t_q - u)^{2H} - (t_{q+1} - t_q)^{2H}}{2H(2H - 1)} \\
&\quad - 2^m(t_{q+1} - t_q)(t_q - u) \frac{(t_{q+1} - t_{q-1})^{2H} - (t_q - t_{q-1})^{2H} - (t_{q+1} - t_q)^{2H}}{2H(2H - 1)}
\end{aligned}$$

Notice that every element in the numerators is smaller or equal than 2^{-2Hm} except $(t_{q+1} - u)^{2H}$ and $(t_{q+1} - t_{q-1})^{2H}$ for which the following relation holds $(t_{q+1} - u)^{2H} \leq (t_{q+1} - t_{q-1})^{2H} = 2^{2H} 2^{-2Hm}$. Therefore, taking absolute value of each element we have

$$\leq \frac{2^{2H} + 2}{H(2H - 1)} 2^{-2Hm}$$

Now let's focus on the second part, that is

$$\begin{aligned}
&\sum_{t_i=t_{q+1}}^{t_{p-1}} \int_{t_i}^{t_{i+1}} \int_u^{t_q} \left(|t-s|^{2H-2} - 2^{2m} \int_{t_i}^{t_{i+1}} \int_{t_{q-1}}^{t_q} |x-y|^{2H-2} dy dx \right) ds dt \\
&= \sum_{t_i=t_{q+1}}^{t_{p-1}} \int_{t_i}^{t_{i+1}} \int_u^{t_q} \left((t-s)^{2H-2} - 2^{2m} \int_{t_i}^{t_{i+1}} \int_{t_{q-1}}^{t_q} (x-y)^{2H-2} dy dx \right) ds dt \\
&= \sum_{t_i=t_{q+1}}^{t_{p-1}} \int_{t_i}^{t_{i+1}} \int_u^{t_q} 2^{2m} \left(\int_{t_i}^{t_{i+1}} \int_{t_{q-1}}^{t_q} (t-s)^{2H-2} - (x-y)^{2H-2} dy dx \right) ds dt
\end{aligned}$$

By applying Mean Value Theorem we have

$$\begin{aligned}
&\leq \sum_{t_i=t_{q+1}}^{t_{p-1}} \int_{t_i}^{t_{i+1}} \int_u^{t_q} 2^{2m} \left(\int_{t_i}^{t_{i+1}} \int_{t_{q-1}}^{t_q} \sup_{\xi \in [(t-s), (x-y)]} (2-2H) \xi^{2H-3} |t-s-(x-y)| dy dx \right) ds dt \\
&= \sum_{t_i=t_{q+1}}^{t_{p-1}} \int_{t_i}^{t_{i+1}} \int_u^{t_q} 2^{2m} \left(\int_{t_i}^{t_{i+1}} \int_{t_{q-1}}^{t_q} (2-2H)(t_i - t_q)^{2H-3} |t-x+y-s| dy dx \right) ds dt
\end{aligned}$$

Notice that for $H = 1$ then the equation above is just zero. Thus, for the following we are in the case $H \in (\frac{1}{2}, 1)$. Notice also that $|t-x| \leq 2^{-m}$ and $|y-s| \leq 2^{-m}$, hence $|t-x+y-s| \leq 2^{1-m}$. Thus, we have

$$\begin{aligned}
&\leq \sum_{t_i=t_{q+1}}^{t_{p-1}} \int_{t_i}^{t_{i+1}} \int_u^{t_q} 2^{2m} \left(\int_{t_i}^{t_{i+1}} \int_{t_{q-1}}^{t_q} (4-4H)(t_i - t_q)^{2H-3} 2^{-m} dy dx \right) ds dt \\
&\leq (4-4H) 2^{-3m} \sum_{t_i=t_{q+1}}^{t_{p-1}} (t_i - t_q)^{2H-3}
\end{aligned}$$

Notice that

$$\begin{aligned}
&\leq (4-4H) 2^{-3m} \sum_{i=1}^{2^m} \left(\frac{i}{2^m} \right)^{2H-3} = (4-4H) 2^{-2Hm} \sum_{i=1}^{2^m} (i)^{2H-3} \leq (4-4H) 2^{-2Hm} \sum_{i=1}^{\infty} (i)^{2H-3} \\
&\leq K 2^{-2Hm}
\end{aligned}$$

where K is a finite constant independent of m . Notice that $\sum_{i=1}^{\infty} (i)^{2H-3} < \infty$ for $H < 1$. However, here we are only in the case $H < 1$ as discussed above.

A similar reasoning applies to the integral $\int_{t_p}^v \int_{t_q}^p$. □

The last integral to be analysed is $\int_{t_p}^v \int_u^{t_q}$.

Proposition 4.3. *The difference*

$$\int_{t_p}^v \int_u^{t_q} \left(|t-s|^{2H-2} - \sum_{i,j=1}^{2^m} \mathbf{1}_{[t_i, t_{i+1}] \times [t_j, t_{j+1}]}(t, s) 2^{2m} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dx dy \right) ds dt$$

is bounded by

$$\frac{3^{2H} + 10(2^{2H}) + 2}{2H(2H-1)} 2^{-2Hm}.$$

Proof. Here we need to distinguish two cases. One in which $|u-v| \leq \frac{2}{2^m}$ and the other in which $|u-v| > \frac{2}{2^m}$.

Let's focus first on the case $|u-v| > \frac{2}{2^m}$.

$$\begin{aligned} & \int_{t_p}^v \int_u^{t_q} \left(|t-s|^{2H-2} - \sum_{i,j=1}^{2^m} \mathbf{1}_{[t_i, t_{i+1}] \times [t_j, t_{j+1}]}(t, s) 2^{2m} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dx dy \right) ds dt \\ &= \int_{t_p}^v \int_u^{t_q} \left(|t-s|^{2H-2} - 2^{2m} \int_{t_p}^{t_{p+1}} \int_{t_{q-1}}^{t_q} |x-y|^{2H-2} dx dy \right) ds dt \\ &= \int_{t_p}^v \int_u^{t_q} 2^{2m} \left(\int_{t_p}^{t_{p+1}} \int_{t_{q-1}}^{t_q} |t-s|^{2H-2} - |x-y|^{2H-2} dx dy \right) ds dt \end{aligned}$$

By Mean Value Theorem we have

$$\begin{aligned} & \leq \int_{t_p}^v \int_u^{t_q} 2^{2m} \left(\int_{t_p}^{t_{p+1}} \int_{t_{q-1}}^{t_q} \sup_{\xi \in [(t-s), (x-y)]} (2-2H) \xi^{2H-3} |t-s-(x-y)| dx dy \right) ds dt \\ &= \int_{t_p}^v \int_u^{t_q} 2^{2m} \left(\int_{t_p}^{t_{p+1}} \int_{t_{q-1}}^{t_q} (2-2H) (t_p-t_q)^{2H-3} |t-x+y-s| dx dy \right) ds dt \end{aligned}$$

Since $|u-v| > \frac{2}{2^m}$ then $t_p - t_q \geq 2^{-m}$.

$$\begin{aligned} & \leq \int_{t_p}^v \int_u^{t_q} 2^{2m} \left(\int_{t_p}^{t_{p+1}} \int_{t_{q-1}}^{t_q} (4-4H) 2^{-2Hm+3m} 2^{-m} dx dy \right) ds dt \\ & \leq (4-4H) 2^{-2Hm} \end{aligned}$$

Finally consider the second case, i.e. $|u-v| \leq \frac{2}{2^m}$.

$$\begin{aligned} & \int_{t_p}^v \int_u^{t_q} \left(|t-s|^{2H-2} - 2^{2m} \int_{t_p}^{t_{p+1}} \int_{t_{q-1}}^{t_q} |x-y|^{2H-2} dx dy \right) ds dt \\ &= \frac{(v-u)^{2H} - (t_p-u)^{2H} - (v-t_q)^{2H} + (t_p-t_q)^{2H}}{2H(2H-1)} \end{aligned}$$

$$-2^{2m}(v-t_p)(t_q-u) \frac{(t_{p+1}-t_{q-1})^{2H} - (t_p-t_{q-1})^{2H} - (t_{p+1}-t_q)^{2H} + (t_p-t_q)^{2H}}{2H(2H-1)}$$

Now since $|u-v| \leq \frac{2}{2^m}$ then $t_p-t_q \leq \frac{1}{2^m}$ so $t_{p+1}-t_{q-1} \leq \frac{3}{2^m}$. Therefore, we have

$$\leq \left(\frac{3^{2H} + 10(2^{2H}) + 2}{2H(2H-1)} \right) 2^{-2Hm}.$$

Moreover, since

$$(4-4H) < \frac{3^{2H} + 10(2^{2H}) + 2}{2H(2H-1)}$$

then we can use the right hand side of the above inequality as the coefficient for the rate of convergence for the integral $\int_{t_p}^v \int_u^{t_q}$ independently of the value of u and v . \square

Therefore, we have the following proposition

Proposition 4.4. *The difference for the double integral $\int_{u < s < t < v}$*

$$\int_{u < s < t < v} \left(|t-s|^{2H-2} - \sum_{i,j=1}^{2^m} \mathbf{1}_{[t_i, t_{i+1}] \times [t_j, t_{j+1}]}(t, s) 2^{2m} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dx dy \right) ds dt$$

is bounded by

$$A 2^{-2Hm}$$

where A is given by

$$A = 2 \left(\frac{1}{H(2H-1)} + \frac{2^{2H} + 2}{H(2H-1)} + (4-4H) 2^{-2Hm} \sum_{i=1}^{\infty} (i)^{2H-3} \right) + \frac{3^{2H} + 10(2^{2H}) + 2}{2H(2H-1)}$$

for $H < 1$, and

$$A = 2 \left(\frac{1}{H(2H-1)} + \frac{2^{2H} + 2}{H(2H-1)} \right) + \frac{3^{2H} + 10(2^{2H}) + 2}{2H(2H-1)}$$

for $H = 1$.

Proof. It is immediate from the previous propositions, from the fact that for the double integral $\int_{t_q}^{t_p} \int_{t_q}^t$ the difference is zero as discussed before and by considering equation (7). Indeed, the difference for the double integral $\int_{t_q}^{t_p} \int_{t_q}^t$ is given by

$$\int_{t_q}^{t_p} \int_{t_q}^t \left(|t-s|^{2H-2} - \sum_{i,j=1}^{2^m} \mathbf{1}_{[t_i, t_{i+1}] \times [t_j, t_{j+1}]}(t, s) 2^{2m} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dx dy \right) ds dt$$

Notice that this difference can be expressed as the sum of two classes of integrals. The first class contains the off-diagonal terms, which can be expressed as

$$\int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} \left(|t-s|^{2H-2} - 2^{2m} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dx dy \right) ds dt.$$

where $t_i, t_j \in \{t_q, t_{q+1}, \dots, t_{p-1}, t_p\}$. It is possible to see that this difference it is immediately zero. The second case is

$$\int_{t_i}^{t_{i+1}} \int_{t_i}^t \left(|t-s|^{2H-2} - 2^{2m} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} |x-y|^{2H-2} dx dy \right) ds dt$$

where $t_i \in \{t_q, t_{q+1}, \dots, t_{p-1}\}$. This difference is equal to

$$\begin{aligned} &= \frac{(t_{i+1} - t_i)^{2H}}{2H(2H-1)} - \int_{t_i}^{t_{i+1}} \int_{t_i}^t 2^{2m} \frac{(t_{i+1} - t_i)^{2H}}{H(2H-1)} ds dt \\ &= \frac{2^{-2Hm}}{2H(2H-1)} - \frac{2^{2m-2Hm}}{H(2H-1)} \int_{t_i}^{t_{i+1}} \int_{t_i}^t ds dt \\ &= \frac{2^{-2Hm}}{2H(2H-1)} - \frac{2^{2m-2Hm}}{H(2H-1)} \frac{2^{-2m}}{2} = 0 \end{aligned}$$

This result could also be seen from the symmetric properties with respect to the diagonal of the function $|t-s|^{2H-2}$. \square

Therefore, we can now do the proof of theorem 2.2.

Proof.

$$\begin{aligned} &\left| \mathbb{E} \left(\int_{\Delta^{2k}[0,1]} dB^I \right) - \mathbb{E} \left(\int_{\Delta^{2k}[0,1]} dB^{m,I} \right) \right| = \\ &= \left| \frac{H^k(2H-1)^k}{k!2^k} \sum_{\sigma \in \mathcal{G}_{2k}} \int_{\Delta^{2k}[0,1]} \prod_{l=1}^k \delta_{i_{\sigma(2l)}, i_{\sigma(2l-1)}} |t_{\sigma(2l)} - t_{\sigma(2l-1)}|^{2H-2} \right. \\ &\quad \left. - \prod_{l=1}^k \delta_{i_{\sigma(2l)}, i_{\sigma(2l-1)}} \sum_{i,j=1}^{2m} \mathbf{1}_{[t_i, t_{i+1}] \times [t_j, t_{j+1}]} (t_{\sigma(2l)}, t_{\sigma(2l-1)}) 2^{2m} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dx dy dt_1 \cdots dt_{2k} \right| \end{aligned}$$

Now by using the relation

$$\prod_{l=1}^k a_l - \prod_{l=1}^k b_l = (a_1 - b_1) \prod_{l=2}^k b_l + a_1(a_2 - b_2) \prod_{l=3}^k b_l + \cdots + \prod_{l=1}^{k-1} a_l (a_k - b_k)$$

where $a_i, b_i \in \mathbb{R}$ for $i = 1, \dots, k$, and by using proposition 4.4, this absolute value is bounded by

$$\begin{aligned} &\leq \left| \frac{A 2^{-2Hm} H^k (2H-1)^k}{k!2^k} \right. \\ &\quad \sum_{\sigma \in \mathcal{G}_{2k}} \int_{\Delta^{2k}[0,1]} \prod_{l=2}^k \delta_{i_{\sigma(2l)}, i_{\sigma(2l-1)}} \sum_{i,j=1}^{2m} \mathbf{1}_{[t_i, t_{i+1}] \times [t_j, t_{j+1}]} (t_{\sigma(2l)}, t_{\sigma(2l-1)}) 2^{2m} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dx dy \\ &\quad \left. + \delta_{i_{\sigma(2)}, i_{\sigma(1)}} |t_{\sigma(2)} - t_{\sigma(1)}|^{2H-2} \prod_{l=3}^k \delta_{i_{\sigma(2l)}, i_{\sigma(2l-1)}} \sum_{i,j=1}^{2m} \mathbf{1}_{[t_i, t_{i+1}] \times [t_j, t_{j+1}]} (t_{\sigma(2l)}, t_{\sigma(2l-1)}) 2^{2m} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dx dy \right| \end{aligned}$$

$$+ \dots + \left| \prod_{l=1}^{k-1} \delta_{i_{\sigma(2l)}, i_{\sigma(2l-1)}} |t_{\sigma(2l)} - t_{\sigma(2l-1)}|^{2H-2} dt_1 \dots dt_{2k} \right| \quad (8)$$

Consider now the single addends in the formula above. It is possible to see that each element is bounded by

$$\int_0^1 \dots \int_0^1 \prod_{l=2}^k |t_{\sigma(2l)} - t_{\sigma(2l-1)}|^{2H-2} dt_1 \dots dt_{2k}$$

The reason why we have $l = 2$ to k in the product does not matter. Indeed, once we take the integral to be over $[0, 1]^k$ instead of Δ^k it is not important which permutation we consider. We can move the integrands around using Tonelli's theorem since they are positive value. Further, the above bound is the same as considering $m = 0$. This integral reduces to

$$= \frac{1}{H^{k-1}(2H-1)^{k-1}}$$

Therefore, we can bound (8) by

$$\left| \frac{A 2^{-2Hm} H^k (2H-1)^k (2k!)}{k! 2^k} \frac{k}{H^{k-1} (2H-1)^{k-1}} \right| \quad (9)$$

where we have used the fact that $\sum_{\sigma \in \mathcal{G}_{2k}}$, which is the sum over all permutations of the set $\{1, \dots, 2k\}$, corresponds to $(2k!)$. Therefore we can rewrite the formula above as

$$= C 2^{-2Hm}$$

where

$$C := \frac{AH(2H-1)(2k!)}{(k-1)! 2^k} < \infty.$$

This concludes the proof of the rate of convergence. □

The sharpness of the bound is a delicate matter. From one side we always used sharp estimates for each value we considered in the proof of the theorem. Further, it has been proved by [15] that for the Brownian motion the sharp rate of convergence is 2^{-m} . Hence, we are confident that the rate of convergence obtained here is sharp. However, in order to be completely sure about it we need to prove that

$$\limsup_{m \rightarrow \infty} 2^{2Hm} \left| \mathbb{E} \left(\int_{\Delta^{2k}[0,1]} dB^I \right) - \mathbb{E} \left(\int_{\Delta^{2k}[0,1]} dB^{m,I} \right) \right| > 0.$$

This will prove sharpness of the rate of convergence. However, the dyadic approximation makes this idea extremely difficult to put in practice. This is because, when we focus on the double integral $\int_{u < s < t < v}$, the dyadic approximation and the exact value for certain times are equal, in particular when the times u and v correspond to the times of the mesh of the dyadic approximation t_q and t_p respectively. Hence as $m \rightarrow \infty$ the difference between the exact value and the dyadic approximation goes to zero discontinuously (or better continuously but with jumps/discontinuous point, i.e. stepwise). This makes the problem of the sharpness of the rate of convergence to appear impossible. A possible solution is to consider the iterated integral as a whole without focusing on the general double integral. But this seems a difficult way of dealing with the

problem. Indeed, we proceeded on with this way at the beginning of this work but the result was a bound with a rate of convergence of 2^{-2Hm} but with a non integrable coefficient. For these reasons we are confident that the rate of convergence of 2^{-2Hm} is sharp.

A final remark regards the fact that $2^{-2Hm} < 2^{-m}$, which means that the difference goes to zero faster for the fBm than the Bm. This is an expected result since the fBm with $H > \frac{1}{2}$ is a smoother path than the Bm. This higher smoothness is related to the positive correlation of the increments of the fBm with $H > \frac{1}{2}$. Indeed, we can see that the rate of convergence becomes smaller (hence the convergence faster) as the positive correlation (represented by H) increases.

5 Problem 2: Bound for the coefficient

As it is possible to see from equation (9) the coefficient of the rate of convergence goes to infinity as the degree of the truncated signature k goes to infinity. In other words, $C \rightarrow \infty$ as $k \rightarrow \infty$. In the following we show that the coefficient in fact goes to zero as k goes to infinity.

In this chapter we prove the theorem 2.3.

Proof. First, multiply our main object of study

$$\left| \mathbb{E} \left(\int_{\Delta^{2k}[0,1]} dB^I \right) - \mathbb{E} \left(\int_{\Delta^{2k}[0,1]} dB^{m,I} \right) \right|$$

by 2^{2Hm} . We can use our estimates obtained in (8) to get:

$$\begin{aligned} & \leq \frac{AH^k(2H-1)^k}{k!2^k} \\ & \sum_{\sigma \in \mathcal{G}_{2k}} \int_{\Delta^{2k}[0,1]} \prod_{l=2}^k \delta_{i_{\sigma(2l)}, i_{\sigma(2l-1)}} \sum_{i,j=1}^{2^m} \mathbf{1}_{[t_i, t_{i+1}] \times [t_j, t_{j+1}]}(t_{\sigma(2l)}, t_{\sigma(2l-1)}) 2^{2m} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dx dy \\ & + \delta_{i_{\sigma(2)}, i_{\sigma(1)}} |t_{\sigma(2)} - t_{\sigma(1)}|^{2H-2} \prod_{l=3}^k \delta_{i_{\sigma(2l)}, i_{\sigma(2l-1)}} \sum_{i,j=1}^{2^m} \mathbf{1}_{[t_i, t_{i+1}] \times [t_j, t_{j+1}]}(t_{\sigma(2l)}, t_{\sigma(2l-1)}) 2^{2m} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |x-y|^{2H-2} dx dy \\ & + \dots + \prod_{l=1}^{k-1} \delta_{i_{\sigma(2l)}, i_{\sigma(2l-1)}} |t_{\sigma(2l)} - t_{\sigma(2l-1)}|^{2H-2} dt_1 \dots dt_{2k} \end{aligned}$$

We have not written it in absolute value because it is a positive value. Now, taking the limit as $m \rightarrow \infty$. By dominated convergence theorem we obtain:

$$\begin{aligned} & \frac{AH^k(2H-1)^k}{k!2^k} \sum_{\sigma \in \mathcal{G}_{2k}} \int_{\Delta^{2k}[0,1]} \prod_{l=2}^k \delta_{i_{\sigma(2l)}, i_{\sigma(2l-1)}} |t_{\sigma(2l)} - t_{\sigma(2l-1)}|^{2H-2} \\ & + \delta_{i_{\sigma(2)}, i_{\sigma(1)}} |t_{\sigma(2)} - t_{\sigma(1)}|^{2H-2} \prod_{l=3}^k \delta_{i_{\sigma(2l)}, i_{\sigma(2l-1)}} |t_{\sigma(2l)} - t_{\sigma(2l-1)}|^{2H-2} \end{aligned}$$

$$+ \dots + \prod_{l=1}^{k-1} \delta_{i_{\sigma(2l)}, i_{\sigma(2l-1)}} |t_{\sigma(2l)} - t_{\sigma(2l-1)}|^{2H-2} dt_1 \cdots dt_{2k} \quad (10)$$

Notice that

$$\begin{aligned} & \sum_{\sigma \in \mathcal{G}_{2k}} \prod_{l=2}^k \delta_{i_{\sigma(2l)}, i_{\sigma(2l-1)}} |t_{\sigma(2l)} - t_{\sigma(2l-1)}|^{2H-2} \\ &= \sum_{\sigma \in \mathcal{G}_{2k}} \delta_{i_{\sigma(2)}, i_{\sigma(1)}} |t_{\sigma(2)} - t_{\sigma(1)}|^{2H-2} \prod_{l=3}^k \delta_{i_{\sigma(2l)}, i_{\sigma(2l-1)}} |t_{\sigma(2l)} - t_{\sigma(2l-1)}|^{2H-2} \\ &= \dots = \sum_{\sigma \in \mathcal{G}_{2k}} \prod_{l=1}^{k-1} \delta_{i_{\sigma(2l)}, i_{\sigma(2l-1)}} |t_{\sigma(2l)} - t_{\sigma(2l-1)}|^{2H-2}. \end{aligned}$$

Moreover notice that the function $f(t_1, \dots, t_{2k})$ defined as

$$f(t_1, \dots, t_{2k}) := \sum_{\sigma \in \mathcal{G}_{2k}} \prod_{l=2}^k \delta_{i_{\sigma(2l)}, i_{\sigma(2l-1)}} |t_{\sigma(2l)} - t_{\sigma(2l-1)}|^{2H-2}$$

is symmetric with respect to the diagonal. Hence we have,

$$\int_{\Delta^{2k}_{[0,1]}} f(t_1, \dots, t_{2k}) dt_1 \cdots dt_{2k} = \frac{1}{2k!} \int_{[0,1]^{2k}} f(t_1, \dots, t_{2k}) dt_1 \cdots dt_{2k}$$

Therefore we have that the formula (10) is equal to

$$\begin{aligned} & \frac{AH^k(2H-1)^k}{k!2^k} \frac{k}{(2k)!} \sum_{\sigma \in \mathcal{G}_{2k}} \int_{[0,1]^{2k}} \prod_{l=2}^k \delta_{i_{\sigma(2l)}, i_{\sigma(2l-1)}} |t_{\sigma(2l)} - t_{\sigma(2l-1)}|^{2H-2} dt_1 \cdots dt_{2k} \\ & \leq \frac{AH^k(2H-1)^k}{k!2^k} \frac{k}{(2k)!} \frac{(2k)!}{H^{k-1}(2H-1)^{k-1}} = \frac{AH(2H-1)}{(k-1)!2^k} \end{aligned}$$

□

Notice that

$$\frac{AH(2H-1)}{(k-1)!2^k} \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty.$$

Thus, the coefficient of the rate of convergence goes to zero as the number of iterated integrals increases (i.e. as the order of the signature increases) and it goes there very fast.

6 Problem 3: Estimate of the expected signature

In this chapter we shift the focus we had in the last two chapters. Our object is now just the expected signature of the fractional Brownian motion (fBm). We prove a simple but precise estimate for it.

Indeed, we prove theorem 2.4. The proof of this theorem is very short since it relies on the computations done in the proof of theorem 2.3.

Proof. Consider the (truncated) signature of the fractional Brownian motion, we have

$$\begin{aligned} \mathbb{E} \left(\int_{\Delta^{2k}[0,1]} dB^I \right) &= \frac{H^k(2H-1)^k}{k!2^k} \sum_{\sigma \in \mathcal{G}_{2k}} \int_{\Delta^{2k}[0,1]} \prod_{l=1}^k \delta_{i_{\sigma(2l)}, i_{\sigma(2l-1)}} |t_{\sigma(2l)} - t_{\sigma(2l-1)}|^{2H-2} dt_1 \cdots dt_{2k}. \\ &\leq \frac{H^k(2H-1)^k}{k!2^k} \frac{1}{2k!} \frac{2k!}{H^k(2H-1)^k} \end{aligned}$$

by the proof of theorem 2.3.

Further, we have

$$\frac{H^k(2H-1)^k}{k!2^k} \frac{1}{2k!} \frac{2k!}{H^k(2H-1)^k} = \frac{1}{k!2^k}$$

□

In the next chapter we shift again our attention and we focus on the cubature method which uses the signature and its approximation in order to obtain numerical solutions of SDEs driven by the fBm.

Indeed, one of the main reasons why we focused on these problems and tried to obtain these results is to provide good rigorous estimates (about the rate of convergence, the coefficient, the signature, etc..) to improve the theoretical capability of the numerical methods (mainly the cubature method) of obtaining good results in an efficient amount of time. This is fundamental since the cubature method can be a cumbersome method in some cases. Hence, having good theoretical estimates tells us the level of performance we can achieve, at least theoretically.

7 Problem 4: Cubature for fractional Brownian motion

In this chapter we obtain the cubature formula for the one dimensional fractional Brownian motion up to degree 5.

As discussed before (see Section 3 and in particular definition 3.6), in order to obtain the cubature formula what we need to do is to find n paths and n weights such that

$$\mathbb{E} \left[\int_{0 < t_1 < \dots < t_k < T} d\hat{B}_{t_1}^{i_1} \cdots d\hat{B}_{t_k}^{i_k} \right] = \sum_{j=1}^n \lambda_j \int_{0 < t_1 < \dots < t_k < T} d\hat{\omega}_{j,t_1}^{i_1} \cdots d\hat{\omega}_{j,t_1}^{i_k}.$$

In our case, since we focus on the one dimensional case, $i_l \in \{0, 1\}$ for $l = 1, \dots, k$. For $i_l = 0$ we mean that \hat{B}_t^0 and $\hat{\omega}_{j,t}^0$ are both the path t (i.e. $\hat{B}_t^0 = \hat{\omega}_{j,t}^0 = t$). On the other hand, for $i_l = 1$ we mean that \hat{B}_t^1 is the fractional Brownian motion at time t and $\hat{\omega}_{j,t}^1$ is the deterministic path (that we need to find) at time t that we denote $\omega_{j,t}$.

Further, since we focus on degree up to 5 then $k \leq 5$. In the Appendix 1 we provide a complete description of the iterated integrals we have to consider for this cubature formula, they are 17 in total. Further, again in the Appendix 1 we provide their values and the corresponding equations for the deterministic paths $\omega_{j,t}$.

In this chapter we should prove theorem 2.5. However, due to the fact that the proof is mainly based on linear algebra computations, we decided to put the proof of this theorem in the Appendix (see Appendix 2).

Now, we need to discuss some important remarks about this result. Given the definition of \mathcal{A}_m (see Remark 3.7) we have that the cubature formula obtained in theorem 2.5 is valid for the following degree

$$\text{Degree} = \begin{cases} 5 & \text{for } \frac{1}{2} \leq H < \frac{2}{3}, \\ 4 & \text{for } \frac{2}{3} \leq H < 1, \\ 3 & \text{for } H = 1. \end{cases}$$

The reason why we have different degrees for different values of H is because we wanted to have the same cubature formula for different values of H . In particular, we wanted a cubature formula which was the same obtained by Lyons and Victoir in [13] for the Brownian motion of degree 5. In order to have it we needed to consider the same iterated integrals they considered.

In other words, we decided to fix the number of iterated integrals and so to fix the system of 17 equations we needed to solve in order to have a cubature formula that was exactly as the one of [13] for $H = \frac{1}{2}$. This had as a counterpart the fact that for high value of H the degree have to be decreased according to the definition of \mathcal{A}_m (see Remark 3.7).

Further, in case we wanted to have a cubature formula for degree 5 (independently of H) then we would had to compute more iterated integrals than the ones we computed. In this way we would add more constraints ending up with a cubature formula different from the one of [13].

Given that the concept of *degree* for a general stochastic process is not usually known, we introduce now a more general concept which is related to the degree and can be adapted to any stochastic process: the concept of *order*.

Definition 7.1. *Let W be a stochastic process. We say that a set of words J is of order l with respect to the stochastic process W if for any $dY_t = V(Y_t)dW_t$, where V is any smooth vector field, we have*

$$\left| \mathbb{E}(f(Y_t)) - \sum_{i_1, \dots, i_k \in J} V_{i_1} \cdots V_{i_k} f(Y_0) \mathbb{E} \left(\int_{0 < s_1 < \dots < s_k < t} dW_{s_1}^{i_1} \cdots dW_{s_k}^{i_k} \right) \right| \leq ct^l$$

$\forall t < \varepsilon$, for ε small, and $\forall f$ smooth and bounded. Where c is a constant.

Further, we introduce the concept of *cubature formula of order l* .

Definition 7.2. *A cubature formula of order l is obtained by matching*

$$\mathbb{E} \left(\int_{0 < s_1 < \dots < s_k < t} dW_{s_1}^{i_1} \cdots dW_{s_k}^{i_k} \right) = \sum_{p=0}^n \lambda_p \int_{0 < s_1 < \dots < s_k < t} d\omega_{s_1}^{p, i_1} \cdots d\omega_{s_k}^{p, i_k}$$

$\forall i_1, \dots, i_k \in J$, for J a set of words of order l with respect to the process W . Where $\lambda_0, \dots, \lambda_n$ are positive weights and $\omega_s^{0, i_1}, \dots, \omega_s^{n, i_k}$ are continuous and deterministic paths.

Using these definitions we can state that the cubature formula that we have obtained in theorem 2.5 is of order $\frac{5}{2}$. This is because the lowest order is obtained for $H = \frac{1}{2}$, which is the Brownian motion case for which we have an order of $\frac{5}{2}$. Indeed, for the fractional Brownian motion we can make the following remark.

Remark 7.3. *For the fractional Brownian motion, and so also for the Brownian motion, the relation between the order and the degree of a cubature formula is given by*

$$\text{Order} = \text{Degree} \times H.$$

where H is the Hurst parameter.

Hence, precisely the order of the cubature formula of theorem 2.5 is

$$\text{Order} = \begin{cases} 5H & \text{for } \frac{1}{2} \leq H < \frac{2}{3}, \\ 4H & \text{for } \frac{2}{3} \leq H < 1, \\ 3H & \text{for } H = 1. \end{cases}$$

We conclude with a final remark on the solution(s) obtained for the cubature formula.

Remark 7.4. *From the proof of this theorem (see Appendix 2) we obtain two solutions of our system of equations. In particular, we have only used one solution of the equation*

$$c_1^2 \frac{1}{27} - c_1 \frac{4\sqrt{3}}{27} = \frac{5 - 8H}{3(2H + 1)}.$$

The other solution is given by

$$c_1 = \frac{4H\sqrt{3} + 2\sqrt{3} + \sqrt{-96H^2 + 66H + 57}}{2H + 1}$$

The reason why we focused on the other solution and not this one is because Lyons and Victoir focused on that solution in their paper. They used MATHEMATICA to produce a solution, without explicitly justify their decision. However, both solutions are feasible for our system of equations (17). Hence, we have two valid solutions for the cubature formula of the fractional Brownian motion for this kind of structure (i.e. piecewise linear paths with change of slopes at $t = \frac{1}{3}$ and $t = \frac{2}{3}$).

The reason why they did not justify their decision is probably due to the fact that the structure adopted is already arbitrary and hence it is important to have a solution and not to have a particular or unique solution.

8 Acknowledgements

The author would like to thank Dr. H. Boedihardjo (University of Reading) and Prof. D. Crisan (Imperial College London) for numerous and useful comments, suggestions and discussions during the whole production of this work. Further, the author would like to thank Dr. T. Kuna (University of Reading) for useful comments regarding Problem 2 and Problem 3 of this work and Dr. J. Broecker (University of Reading) for different and useful discussions. Finally, the author would like to thank the Centre for Doctoral Training in Mathematics of Planet Earth of the University of Reading and of the Imperial College London for providing funding for this research.

9 Appendix 1: Iterated integrals for the cubature method

In this appendix we study the iterated integrals with respect to the path (t, B_t^H) . Notice that we will use the notation $B_t := B_t^H$. We will not consider the case of iterated integrals of time solely since they bring no information for the construction of the cubature. Further, we will use many times the following formula for the fractional Brownian motion:

$$\mathbb{E} \left((B_t - B_s)^{2k} \right) = \frac{(2k!)}{k!2^k} |t - s|^{2Hk}$$

In particular, we have:

Degree= 1:

$$\mathbb{E} \left(\int_0^1 dB_{u_1} \right) = 0$$

Degree= 2:

$$\mathbb{E} \left(\int_0^1 \int_0^{u_2} dB_{u_1} dB_{u_2} \right) = \mathbb{E} \left(\frac{B_1^2}{2} \right) = \frac{1}{2}$$

Degree= $1 + \frac{1}{H}$:

$$\mathbb{E} \left(\int_0^1 \int_0^{u_2} dB_{u_1} du_2 \right) = \int_0^1 \mathbb{E}(B_{u_2}) du_2 = 0$$

and

$$\mathbb{E} \left(\int_0^1 \int_0^{u_2} du_1 dB_{u_2} \right) = \mathbb{E} \left(\int_0^1 u_2 dB_{u_2} \right) = 0$$

Degree= 3:

$$\mathbb{E} \left(\int_0^1 \int_0^{u_3} \int_0^{u_2} dB_{u_1} dB_{u_2} dB_{u_3} \right) = \mathbb{E} \left(\frac{B_1^3}{3!} \right) = 0$$

Degree= $2 + \frac{1}{H}$:

$$\mathbb{E} \left(\int_0^1 \int_0^{u_3} \int_0^{u_2} dB_{u_1} dB_{u_2} du_3 \right) = \int_0^1 \mathbb{E} \left(\frac{B_{u_3}^2}{2} \right) du_3 = \int_0^1 \frac{u_3^{2H}}{2} du_3 = \frac{1}{2(2H+1)}$$

and

$$\begin{aligned} \mathbb{E} \left(\int_0^1 \int_0^{u_3} \int_0^{u_2} dB_{u_1} du_2 dB_{u_3} \right) &= \mathbb{E} \left(\int_0^1 \int_0^{u_3} B_{u_2} du_2 dB_{u_3} \right) = \mathbb{E} \left(\int_0^1 \int_{u_2}^1 B_{u_2} dB_{u_3} du_2 \right) \\ &= \int_0^1 \mathbb{E}(B_{u_2}(B_1 - B_{u_2})) du_3 = \int_0^1 \frac{1}{2} (1 - u_2^{2H} - (1 - u_2)^{2H}) du_2 = \frac{1}{2} - \frac{2}{2(2H+1)} = \frac{2H-1}{2(2H+1)} \end{aligned}$$

and

$$\mathbb{E} \left(\int_0^1 \int_0^{u_3} \int_0^{u_2} du_1 dB_{u_2} dB_{u_3} \right) = \mathbb{E} \left(\int_0^1 \int_0^{u_3} \int_{u_1}^{u_3} dB_{u_2} du_1 dB_{u_3} \right) = \mathbb{E} \left(\int_0^1 \int_0^{u_3} B_{u_3} - B_{u_1} du_1 dB_{u_3} \right)$$

$$\begin{aligned}
&= \mathbb{E} \left(\int_0^1 \int_{u_1}^1 B_{u_3} - B_{u_1} dB_{u_3} du_1 \right) = \int_0^1 \mathbb{E} \left(\frac{B_1^2}{2} - \frac{B_{u_1}^2}{2} - B_{u_1}(B_1 - B_{u_1}) \right) du_1 \\
&= \int_0^1 \mathbb{E} \left(\frac{B_1^2}{2} + \frac{B_{u_1}^2}{2} - B_{u_1}B_1 \right) du_1 = \int_0^1 \frac{1}{2} + \frac{u_1^{2H}}{2} - \frac{1}{2}(1 + u_1^{2H} - (1 - u_1)^{2H}) du_1 \\
&= \frac{1}{2} \int_0^1 (1 - u_1)^{2H} du_1 = \frac{1}{2(2H + 1)}
\end{aligned}$$

Degree= 4:

$$\mathbb{E} \left(\int_0^1 \int_0^{u_4} \int_0^{u_3} \int_0^{u_2} dB_{u_1} dB_2 dB_3 dB_4 \right) = \mathbb{E} \left(\frac{B_1^4}{4!} \right) = \frac{1}{8}$$

Degree= $1 + \frac{2}{H}$:

$$\mathbb{E} \left(\int_0^1 \int_0^{u_3} \int_0^{u_2} dB_{u_1} du_2 du_3 \right) = \int_0^1 \int_0^{u_3} \mathbb{E}(B_{u_2}) du_2 du_3 = 0$$

and

$$\mathbb{E} \left(\int_0^1 \int_0^{u_3} \int_0^{u_2} du_1 dB_{u_2} du_3 \right) = \mathbb{E} \left(\int_0^1 \int_0^{u_3} \int_{u_1}^{u_3} dB_{u_2} du_1 du_3 \right) = \int_0^1 \int_0^{u_3} \mathbb{E}(B_{u_3} - B_{u_1}) du_1 du_3 = 0$$

and

$$\mathbb{E} \left(\int_0^1 \int_0^{u_3} \int_0^{u_2} du_1 du_2 dB_{u_3} \right) = \mathbb{E} \left(\int_0^1 \frac{u_3^2}{2} dB_{u_3} \right) = 0$$

Degree= $3 + \frac{1}{H}$:

$$\mathbb{E} \left(\int_0^1 \int_0^{u_4} \int_0^{u_3} \int_0^{u_2} dB_{u_1} dB_{u_2} dB_{u_3} du_4 \right) = \int_0^1 \mathbb{E} \left(\frac{B_{u_4}^3}{3!} \right) du_4 = 0$$

and

$$\begin{aligned}
\mathbb{E} \left(\int_0^1 \int_0^{u_4} \int_0^{u_3} \int_0^{u_2} dB_{u_1} dB_{u_2} du_3 dB_{u_4} \right) &= \mathbb{E} \left(\int_0^1 \int_0^{u_4} \frac{B_{u_3}^2}{2!} du_3 dB_{u_4} \right) = \mathbb{E} \left(\int_0^1 \int_{u_3}^1 \frac{B_{u_3}^2}{2!} dB_{u_4} du_3 \right) \\
&= \int_0^1 \mathbb{E} \left((B_1 - B_{u_3}) \frac{B_{u_3}^2}{2!} \right) du_3 = 0
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E} \left(\int_0^1 \int_0^{u_4} \int_0^{u_3} \int_0^{u_2} dB_{u_1} du_2 dB_{u_3} dB_{u_4} \right) = \mathbb{E} \left(\int_0^1 \int_0^{u_4} \int_0^{u_3} B_{u_2} du_2 dB_{u_3} dB_{u_4} \right) \\
&= \mathbb{E} \left(\int_0^1 \int_0^{u_4} \int_{u_2}^{u_4} B_{u_2} dB_{u_3} du_2 dB_{u_4} \right) = \mathbb{E} \left(\int_0^1 \int_0^{u_4} (B_{u_4} - B_{u_2}) B_{u_2} du_2 dB_{u_4} \right) \\
&= \mathbb{E} \left(\int_0^1 \int_{u_2}^1 (B_{u_4} - B_{u_2}) B_{u_2} dB_{u_4} du_2 \right) = \int_0^1 \mathbb{E} \left(\frac{B_{u_2}}{2} (B_1 - B_{u_2})^2 - B_{u_2}^2 (B_1 - B_{u_2}) \right) du_2 = 0
\end{aligned}$$

and

$$\begin{aligned}
& \mathbb{E} \left(\int_0^1 \int_0^{u_4} \int_0^{u_3} \int_0^{u_2} du_1 dB_{u_2} dB_{u_3} dB_{u_4} \right) = \mathbb{E} \left(\int_0^1 \int_0^{u_4} \int_0^{u_3} \int_{u_1}^{u_3} dB_{u_2} du_1 dB_{u_3} dB_{u_4} \right) \\
& = \mathbb{E} \left(\int_0^1 \int_0^{u_4} \int_0^{u_3} (B_{u_3} - B_{u_1}) du_1 dB_{u_3} dB_{u_4} \right) = \mathbb{E} \left(\int_0^1 \int_0^{u_4} \int_{u_1}^{u_4} (B_{u_3} - B_{u_1}) dB_{u_3} du_1 dB_{u_4} \right) \\
& = \mathbb{E} \left(\int_0^1 \int_0^{u_4} \frac{B_{u_4}^2}{2} - \frac{B_{u_1}^2}{2} - B_{u_1} (B_{u_4} - B_{u_1}) du_1 dB_{u_4} \right) = \mathbb{E} \left(\int_0^1 \int_{u_1}^1 \frac{B_{u_4}^2}{2} - \frac{B_{u_1}^2}{2} - B_{u_1} (B_{u_4} - B_{u_1}) dB_{u_4} du_1 \right) \\
& = \int_0^1 \mathbb{E} \left(\frac{B_1^3}{3!} - \frac{B_{u_1}^3}{3!} - \frac{B_{u_1}^2}{2} (B_1 - B_{u_1}) - B_{u_1} \left(\frac{B_1^2}{2} - \frac{B_{u_1}^2}{2} \right) + B_{u_1}^2 (B_1 - B_{u_1}) \right) du_1 = 0
\end{aligned}$$

Degree= 5:

$$\mathbb{E} \left(\int_0^1 \int_0^{u_5} \int_0^{u_4} \int_0^{u_3} \int_0^{u_2} dB_{u_1} dB_{u_2} dB_{u_3} dB_{u_4} dB_{u_5} \right) = \mathbb{E} \left(\frac{B_1^5}{5!} \right) = 0$$

Now we need to match them with the corresponding deterministic iterated integrals. In other words, we have

For degree= 1:

$$0 = \sum_{i=1}^n \lambda_i \int_0^1 d\omega_{u_1, i} \Rightarrow \sum_{i=1}^n \lambda_i \omega_{i,1} = 0$$

For degree= 2:

$$\frac{1}{2} = \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_2} d\omega_{i, u_1} d\omega_{i, u_2} \Rightarrow \frac{1}{2} = \frac{1}{2} \sum_{i=1}^n \lambda_i \omega_{i,1}^2 \Rightarrow \sum_{i=1}^n \lambda_i \omega_{i,1}^2 = 1$$

For degree= $1 + \frac{1}{H}$:

$$0 = \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_2} d\omega_{i, u_1} du_2 \Rightarrow \sum_{i=1}^n \lambda_i \int_0^1 \omega_{i, u_2} du_2 = 0$$

and

$$0 = \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_2} du_1 d\omega_{i, u_2} \Rightarrow \sum_{i=1}^n \lambda_i \int_0^1 u_2 d\omega_{i, u_2} = 0$$

For degree= 3:

$$0 = \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_3} \int_0^{u_2} d\omega_{i, u_1} d\omega_{i, u_2} d\omega_{i, u_3} \Rightarrow \frac{1}{3!} \sum_{i=1}^n \lambda_i \omega_{i,1}^3 = 0 \Rightarrow \sum_{i=1}^n \lambda_i \omega_{i,1}^3 = 0$$

For degree= $2 + \frac{1}{H}$:

$$\frac{1}{2(2H+1)} = \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_3} \int_0^{u_2} d\omega_{i, u_1} d\omega_{i, u_2} du_3 \Rightarrow \sum_{i=1}^n \lambda_i \int_0^1 \omega_{i, u_3}^2 du_3 = \frac{1}{2H+1}$$

and

$$\frac{2H-1}{2(2H+1)} = \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_3} \int_0^{u_2} d\omega_{i,u_1} du_2 d\omega_{i,u_3} \Rightarrow \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_3} \omega_{i,u_2} du_2 d\omega_{i,u_3} = \frac{2H-1}{2(2H+1)}$$

and

$$\frac{1}{2(2H+1)} = \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_3} \int_0^{u_2} du_1 d\omega_{i,u_2} d\omega_{i,u_3} \Rightarrow \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_3} u_2 d\omega_{i,u_2} d\omega_{i,u_3} = \frac{1}{2(2H+1)}$$

For degree= 4:

$$\frac{1}{8} = \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_4} \int_0^{u_3} \int_0^{u_2} d\omega_{i,u_1} d\omega_{i,u_2} d\omega_{i,u_3} d\omega_{i,u_4} \Rightarrow \frac{1}{4!} \sum_{i=1}^n \lambda_i \omega_{i,1}^4 = \frac{1}{8} \Rightarrow \sum_{i=1}^n \lambda_i \omega_{i,1}^4 = 3$$

For degree= $1 + \frac{2}{H}$:

$$0 = \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_3} \int_0^{u_2} d\omega_{i,u_1} du_2 du_3 \Rightarrow \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_3} \omega_{i,u_2} du_2 du_3 = 0$$

and

$$0 = \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_3} \int_0^{u_2} du_1 d\omega_{i,u_2} du_3 \Rightarrow \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_3} u_2 d\omega_{i,u_2} du_3 = 0$$

and

$$0 = \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_3} \int_0^{u_2} du_1 du_2 d\omega_{i,u_3} \Rightarrow \sum_{i=1}^n \lambda_i \int_0^1 \frac{u_3^2}{2} d\omega_{i,u_3} = 0 \Rightarrow \sum_{i=1}^n \lambda_i \int_0^1 u_3^2 d\omega_{i,u_3} = 0$$

For degree= $3 + \frac{1}{H}$:

$$0 = \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_4} \int_0^{u_3} \int_0^{u_2} d\omega_{i,u_1} d\omega_{i,u_2} d\omega_{i,u_3} du_4 \Rightarrow \frac{1}{3!} \sum_{i=1}^n \lambda_i \int_0^1 \omega_{i,u_4}^3 du_4 = 0 \Rightarrow \sum_{i=1}^n \lambda_i \int_0^1 \omega_{i,u_4}^3 du_4 = 0$$

and

$$\begin{aligned} 0 &= \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_4} \int_0^{u_3} \int_0^{u_2} d\omega_{i,u_1} d\omega_{i,u_2} du_3 d\omega_{i,u_4} \Rightarrow \frac{1}{2} \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_4} \omega_{i,u_3}^2 du_3 d\omega_{i,u_4} = 0 \\ &\Rightarrow \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_4} \omega_{i,u_3}^3 du_3 d\omega_{i,u_4} = 0 \end{aligned}$$

and

$$0 = \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_4} \int_0^{u_3} \int_0^{u_2} d\omega_{i,u_1} du_2 d\omega_{i,u_3} d\omega_{i,u_4} \Rightarrow \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_4} \int_0^{u_3} \omega_{i,u_2} du_2 d\omega_{i,u_3} d\omega_{i,u_4} = 0$$

and

$$0 = \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_4} \int_0^{u_3} \int_0^{u_2} du_1 d\omega_{i,u_2} d\omega_{i,u_3} d\omega_{i,u_4} \Rightarrow \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_4} \int_0^{u_3} u_2 d\omega_{i,u_2} d\omega_{i,u_3} d\omega_{i,u_4} = 0$$

For degree= 5:

$$0 = \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_5} \int_0^{u_4} \int_0^{u_3} \int_0^{u_2} d\omega_{i,u_1} d\omega_{i,u_2} d\omega_{i,u_3} d\omega_{i,u_4} d\omega_{i,u_5} \Rightarrow \frac{1}{5!} \sum_{i=1}^n \lambda_i \omega_{i,1}^5 = 0 \Rightarrow \sum_{i=1}^n \lambda_i \omega_{i,1}^5 = 0$$

10 Appendix 2: Proof of Theorem 2.5

In this appendix we present the proof of **Theorem 2.5**.

Proof. Let us now start to investigate the form of the functions ω_j for $j = 1, \dots, n$.

From the Appendix it is possible to see that we have 17 equations, hence we need to have 17 unknowns.

Following the work done by Lyons and Victoir we are going to choose two symmetric paths and one path which has constant value zero. This reduces the number of equations to 5. Hence assume that there are two continuous functions, $\omega_{1,s}$ and $\omega_{2,s}$, with the property that $\omega_{1,s} = -\omega_{2,s}$ for $s \in [0, 1]$. Further assume that there is a third path $\omega_{3,s} = 0$ for $s \in [0, 1]$. With this formulation only 5 equations need to be taken into consideration since the other 12 are already satisfied. The 5 equations are the following. First,

$$\frac{1}{2} = \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_2} d\omega_{i,u_1} d\omega_{i,u_2} \Rightarrow \frac{1}{2} = \frac{1}{2} \sum_{i=1}^n \lambda_i \omega_{i,1}^2 \Rightarrow \sum_{i=1}^n \lambda_i \omega_{i,1}^2 = 1 \quad (11)$$

Second,

$$\frac{1}{2(2H+1)} = \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_3} \int_0^{u_2} d\omega_{i,u_1} d\omega_{i,u_2} du_3 \Rightarrow \sum_{i=1}^n \lambda_i \int_0^1 \omega_{i,u_3}^2 du_3 = \frac{1}{2H+1} \quad (12)$$

Third,

$$\frac{2H-1}{2(2H+1)} = \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_3} \int_0^{u_2} d\omega_{i,u_1} du_2 d\omega_{i,u_3} \Rightarrow \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_3} \omega_{i,u_2} du_2 d\omega_{i,u_3} = \frac{2H-1}{2(2H+1)} \quad (13)$$

Fourth,

$$\frac{1}{2(2H+1)} = \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_3} \int_0^{u_2} du_1 d\omega_{i,u_2} d\omega_{i,u_3} \Rightarrow \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_3} u_2 d\omega_{i,u_2} d\omega_{i,u_3} = \frac{1}{2(2H+1)} \quad (14)$$

Fifth,

$$\frac{1}{8} = \sum_{i=1}^n \lambda_i \int_0^1 \int_0^{u_4} \int_0^{u_3} \int_0^{u_2} d\omega_{i,u_1} d\omega_{i,u_2} d\omega_{i,u_3} d\omega_{i,u_4} \Rightarrow \frac{1}{4!} \sum_{i=1}^n \lambda_i \omega_{i,1}^4 = \frac{1}{8} \Rightarrow \sum_{i=1}^n \lambda_i \omega_{i,1}^4 = 3 \quad (15)$$

The other 12 equations are automatically zero by the symmetric properties of the $\omega_{1,s}$ and $\omega_{s,2}$, and by the fact that $\omega_{3,s} = 0$, since the 12 equations involve odd integrals of the ω s.

Now we need to have maximum 5 unknowns in order to solve the system of equations. We have actually have 6 equations since the sum of the weights $\sum_{i=1}^3 \lambda_i = 1$, because we are considering a probability measure. Assume that

$$\omega_{1,s} = \begin{cases} as & \text{for } s \in [0, \frac{1}{3}], \\ b_1 s + b_0 & \text{for } s \in [\frac{1}{3}, \frac{2}{3}], \\ c_1 s + c_0 & \text{for } s \in [\frac{2}{3}, 1]. \end{cases}$$

With this formulation we have 5 unknowns which are a, b_1, c_1, λ_1 and λ_3 . The reason why b_0 and c_0 are not unknowns is because they have to take certain values in order to make the path $\omega_{1,s}$ continuous. Notice that λ_2 is not an unknowns since $\lambda_2 = \lambda_1$. We have 5 unknowns for 6 equations there is a risk that the system cannot be solved. However, we hope that two of the 5 equations are the same. An alternative approach is to

let the points where the slope changes, which we fixed to be at $\frac{1}{3}$ and $\frac{2}{3}$, be two unknowns. Let us now solve the system. Consider equation (11), we have

$$\lambda_1 \omega_{1,1}^2 + \lambda_2 \omega_{2,1}^2 = 1 \Rightarrow 2\lambda_1 \omega_{1,1}^2 = 1 \Rightarrow 2\lambda_1 (c_1 + c_0)^2 = 1$$

Consider equation (12), we have

$$\begin{aligned} 2\lambda_1 \int_0^1 \omega_{1,u_3}^2 du_3 &= \frac{1}{2H+1} \Rightarrow \int_0^{\frac{1}{3}} a^2 u_3^2 du_3 + \int_{\frac{1}{3}}^{\frac{2}{3}} (b_1 u_3 + b_0)^2 du_3 + \int_{\frac{2}{3}}^1 (c_1 u_3 + c_0)^2 du_3 = \frac{1}{2\lambda_1(2H+1)} \\ &\Rightarrow \frac{a^2}{81} + \frac{1}{3b_1} \left(\frac{8}{27} b_1^3 + b_0^3 + \frac{4}{3} b_1^2 b_0 + 2b_1 b_0^2 \right) - \frac{1}{3b_1} \left(\frac{1}{27} b_1^3 + b_0^3 + \frac{1}{3} b_1^2 b_0 + b_1 b_0^2 \right) \\ &+ \frac{1}{3c_1} (c_1^3 + c_0^3 + 3c_1^2 c_0 + 3c_1 c_0^2) - \frac{1}{3c_1} \left(\frac{8}{27} c_1^3 + c_0^3 + \frac{4}{3} c_1^2 c_0 + 2c_1 c_0^2 \right) = \frac{1}{2\lambda_1(2H+1)} \\ &\Rightarrow \frac{a^2}{81} + \frac{1}{3} \left(\frac{7}{27} b_1^2 + b_1 b_0 + b_0^2 \right) + \frac{1}{3} \left(\frac{19}{27} c_1^2 + \frac{5}{3} c_1 c_0 + c_0^2 \right) = \frac{1}{2\lambda_1(2H+1)} \end{aligned} \quad (16)$$

Consider now equation (14), we have

$$\int_0^1 \int_0^{u_3} \int_0^{u_2} du_1 d\omega_{1,u_2} d\omega_{1,u_3} = \frac{1}{4\lambda_1(2H+1)}$$

By Fubini's theorem we have

$$\begin{aligned} \int_0^1 \int_0^{u_3} \int_{u_1}^{u_3} d\omega_{1,u_2} du_1 d\omega_{1,u_3} &= \frac{1}{4\lambda_1(2H+1)} \Rightarrow \int_0^1 \int_0^{u_3} (\omega_{1,u_3} - \omega_{1,u_1}) du_1 d\omega_{1,u_3} = \frac{1}{4\lambda_1(2H+1)} \\ \Rightarrow \int_0^1 \int_{u_1}^1 (\omega_{1,u_3} - \omega_{1,u_1}) d\omega_{1,u_3} du_1 &= \frac{1}{4\lambda_1(2H+1)} \Rightarrow \int_0^1 \frac{\omega_{1,1}^2}{2} - \frac{\omega_{1,u_1}^2}{2} - \omega_{1,u_1} (\omega_{1,1} - \omega_{1,u_1}) du_1 = \frac{1}{4\lambda_1(2H+1)} \\ &\Rightarrow \int_0^1 \frac{\omega_{1,1}^2}{2} + \frac{\omega_{1,u_1}^2}{2} - \omega_{1,1} \omega_{1,u_1} du_1 = \frac{1}{4\lambda_1(2H+1)} \\ &\Rightarrow (c_1 + c_0)^2 - 2(c_1 + c_0) \left[\int_0^{\frac{1}{3}} a u_1 du_1 + \int_{\frac{1}{3}}^{\frac{2}{3}} b_1 u_1 + b_0 du_1 + \int_{\frac{2}{3}}^1 c_1 u_1 + c_0 du_1 \right] \\ &+ \int_0^{\frac{1}{3}} a^2 u_1^2 du_1 + \int_{\frac{1}{3}}^{\frac{2}{3}} (b_1 u_1 + b_0)^2 du_1 + \int_{\frac{2}{3}}^1 (c_1 u_1 + c_0)^2 du_1 = \frac{1}{2\lambda_1(2H+1)} \\ &\Rightarrow (c_1 + c_0)^2 - 2(c_1 + c_0) \left[a \frac{1}{18} + b_1 \frac{1}{6} + b_0 \frac{1}{3} + c_1 \frac{5}{18} + c_0 \frac{1}{3} \right] \\ &+ \frac{a^2}{81} + \frac{1}{3} \left(\frac{7}{27} b_1^2 + b_1 b_0 + b_0^2 \right) + \frac{1}{3} \left(\frac{19}{27} c_1^2 + \frac{5}{3} c_1 c_0 + c_0^2 \right) = \frac{1}{2\lambda_1(2H+1)} \\ &\Rightarrow \frac{a^2}{81} - 2(c_1 + c_0) \left(a \frac{1}{18} + b_1 \frac{1}{6} + b_0 \frac{1}{3} \right) + \frac{1}{3} \left(\frac{7}{27} b_1^2 + b_1 b_0 + b_0^2 \right) \end{aligned}$$

$$\begin{aligned}
& +c_1^2 \left(1 - \frac{5}{9} + \frac{19}{81}\right) + c_0^2 \left(1 - \frac{2}{3} + \frac{1}{3}\right) + c_1 c_0 \left(2 - \frac{2}{3} - \frac{5}{9} + \frac{5}{9}\right) = \frac{1}{2\lambda_1(2H+1)} \\
\Rightarrow \frac{a^2}{81} - 2(c_1 + c_0) \left(a \frac{1}{18} + b_1 \frac{1}{6} + b_0 \frac{1}{3}\right) + \frac{1}{3} \left(\frac{7}{27}b_1^2 + b_1 b_0 + b_0^2\right) + \frac{55c_1^2}{81} + \frac{2c_0^2}{3} + \frac{4c_1 c_0}{3} = \frac{1}{2\lambda_1(2H+1)}
\end{aligned}$$

Consider now equation (13), we have

$$\int_0^1 \int_0^{u_3} \omega_{1,u_2} du_2 d\omega_{1,u_3} = \frac{2H-1}{4\lambda_1(2H+1)}$$

By Fubini's theorem we have

$$\begin{aligned}
\int_0^1 \int_{u_2}^1 \omega_{1,u_2} d\omega_{1,u_3} du_2 &= \frac{2H-1}{4\lambda_1(2H+1)} \Rightarrow \int_0^1 \omega_{1,u_2} (\omega_{1,1} - \omega_{1,u_2}) du_2 = \frac{2H-1}{4\lambda_1(2H+1)} \\
&\Rightarrow (c_1 + c_0) \left[\int_0^{\frac{1}{3}} a u_1 du_1 + \int_{\frac{1}{3}}^{\frac{2}{3}} b_1 u_1 + b_0 du_1 + \int_{\frac{2}{3}}^1 c_1 u_1 + c_0 du_1 \right] \\
&- \int_0^{\frac{1}{3}} a^2 u_1^2 du_1 - \int_{\frac{1}{3}}^{\frac{2}{3}} (b_1 u_1 + b_0)^2 du_1 - \int_{\frac{2}{3}}^1 (c_1 u_1 + c_0)^2 du_1 = \frac{2H-1}{4\lambda_1(2H+1)} \\
&\Rightarrow (c_1 + c_0) \left[a \frac{1}{18} + b_1 \frac{1}{6} + b_0 \frac{1}{3} + c_1 \frac{5}{18} + c_0 \frac{1}{3} \right] \\
&- \frac{a^2}{81} - \frac{1}{3} \left(\frac{7}{27} b_1^2 + b_1 b_0 + b_0^2 \right) - \frac{1}{3} \left(\frac{19}{27} c_1^2 + \frac{5}{3} c_1 c_0 + c_0^2 \right) = \frac{2H-1}{4\lambda_1(2H+1)} \\
&\Rightarrow (c_1 + c_0) \left(a \frac{1}{18} + b_1 \frac{1}{6} + b_0 \frac{1}{3} \right) - \frac{a^2}{81} - \frac{1}{3} \left(\frac{7}{27} b_1^2 + b_1 b_0 + b_0^2 \right) \\
&+ c_1^2 \left(\frac{5}{18} - \frac{19}{81} \right) + c_0^2 \left(\frac{1}{3} - \frac{1}{3} \right) + c_1 c_0 \left(\frac{1}{3} + \frac{5}{18} - \frac{5}{9} \right) = \frac{2H-1}{4\lambda_1(2H+1)} \\
\Rightarrow (c_1 + c_0) \left(a \frac{1}{18} + b_1 \frac{1}{6} + b_0 \frac{1}{3} \right) - \frac{a^2}{81} - \frac{1}{3} \left(\frac{7}{27} b_1^2 + b_1 b_0 + b_0^2 \right) + \frac{7c_1^2}{162} + \frac{c_1 c_0}{18} &= \frac{2H-1}{4\lambda_1(2H+1)}
\end{aligned}$$

Consider now equation (15), we have

$$\lambda_1 \omega_{1,1}^4 + \lambda_2 \omega_{2,1}^4 = 3 \Rightarrow 2\lambda_1 \omega_{1,1}^4 = 3 \Rightarrow \lambda_1 (c_1 + c_0)^4 = \frac{3}{2}$$

Therefore we have the following system of equations

$$\begin{cases} 2\lambda_1 + \lambda_3 = 1 & \text{with } \lambda_1, \lambda_3 \in [0, 1], \\ 2\lambda_1(c_1 + c_0)^2 = 1, \\ \frac{a^2}{81} + \frac{1}{3} \left(\frac{7}{27}b_1^2 + b_1b_0 + b_0^2 \right) + \frac{1}{3} \left(\frac{19}{27}c_1^2 + \frac{5}{3}c_1c_0 + c_0^2 \right) = \frac{1}{2\lambda_1(2H+1)}, \\ \frac{a^2}{81} - 2(c_1 + c_0) \left(a\frac{1}{18} + b_1\frac{1}{6} + b_0\frac{1}{3} \right) + \frac{1}{3} \left(\frac{7}{27}b_1^2 + b_1b_0 + b_0^2 \right) + \frac{55c_1^2}{81} + \frac{2c_0^2}{3} + \frac{4c_1c_0}{3} = \frac{1}{2\lambda_1(2H+1)}, \\ (c_1 + c_0) \left(a\frac{1}{18} + b_1\frac{1}{6} + b_0\frac{1}{3} \right) - \frac{a^2}{81} - \frac{1}{3} \left(\frac{7}{27}b_1^2 + b_1b_0 + b_0^2 \right) + \frac{7c_1^2}{162} + \frac{c_1c_0}{18} = \frac{2H-1}{4\lambda_1(2H+1)}, \\ \lambda_1(c_1 + c_0)^4 = \frac{3}{2}, \end{cases} \quad (17)$$

with the following unknowns $\lambda_1, \lambda_3, a, b_1, c_1$.

Consider the two equations in the system above

$$\lambda_1(c_1 + c_0)^4 = \frac{3}{2} \quad \text{and} \quad 2\lambda_1(c_1 + c_0)^2 = 1$$

we have

$$\Rightarrow \lambda_1 \frac{1}{4\lambda_1^2} = \frac{3}{2} \Rightarrow \lambda_1 = \frac{1}{6}.$$

and

$$\Rightarrow c_1 + c_0 = \sqrt{3} \quad (18)$$

Further, using $2\lambda_1 + \lambda_3 = 1$ we have

$$\Rightarrow \lambda_3 = \frac{2}{3}$$

Now consider the equations

$$\frac{a^2}{81} - 2(c_1 + c_0) \left(a\frac{1}{18} + b_1\frac{1}{6} + b_0\frac{1}{3} \right) + \frac{1}{3} \left(\frac{7}{27}b_1^2 + b_1b_0 + b_0^2 \right) + \frac{55c_1^2}{81} + \frac{2c_0^2}{3} + \frac{4c_1c_0}{3} = \frac{1}{2\lambda_1(2H+1)}$$

and

$$(c_1 + c_0) \left(a\frac{1}{18} + b_1\frac{1}{6} + b_0\frac{1}{3} \right) - \frac{a^2}{81} - \frac{1}{3} \left(\frac{7}{27}b_1^2 + b_1b_0 + b_0^2 \right) + \frac{7c_1^2}{162} + \frac{c_1c_0}{18} = \frac{2H-1}{4\lambda_1(2H+1)}.$$

By summing them, we have

$$\begin{aligned} & -(c_1 + c_0) \left(a\frac{1}{18} + b_1\frac{1}{6} + b_0\frac{1}{3} \right) + \frac{7c_1^2}{162} + \frac{c_1c_0}{18} + \frac{55c_1^2}{81} + \frac{2c_0^2}{3} + \frac{4c_1c_0}{3} = \frac{3}{2}. \\ & \Rightarrow -(c_1 + c_0) \left(a\frac{1}{18} + b_1\frac{1}{6} + b_0\frac{1}{3} \right) + \frac{13c_1^2}{18} + \frac{25c_1c_0}{18} + \frac{2c_0^2}{3} = \frac{3}{2}. \end{aligned} \quad (19)$$

Further, by taking the difference of the two equations

$$\frac{a^2}{81} + \frac{1}{3} \left(\frac{7}{27} b_1^2 + b_1 b_0 + b_0^2 \right) + \frac{1}{3} \left(\frac{19}{27} c_1^2 + \frac{5}{3} c_1 c_0 + c_0^2 \right) = \frac{1}{2\lambda_1(2H+1)}$$

and

$$\frac{a^2}{81} - 2(c_1 + c_0) \left(a \frac{1}{18} + b_1 \frac{1}{6} + b_0 \frac{1}{3} \right) + \frac{1}{3} \left(\frac{7}{27} b_1^2 + b_1 b_0 + b_0^2 \right) + \frac{55c_1^2}{81} + \frac{2c_0^2}{3} + \frac{4c_1 c_0}{3} = \frac{1}{2\lambda_1(2H+1)}$$

we obtain

$$\begin{aligned} 2(c_1 + c_0) \left(a \frac{1}{18} + b_1 \frac{1}{6} + b_0 \frac{1}{3} \right) + \frac{1}{3} \left(\frac{19}{27} c_1^2 + \frac{5}{3} c_1 c_0 + c_0^2 \right) - \frac{55c_1^2}{81} - \frac{2c_0^2}{3} - \frac{4c_1 c_0}{3} &= 0 \\ \Rightarrow 2(c_1 + c_0) \left(a \frac{1}{18} + b_1 \frac{1}{6} + b_0 \frac{1}{3} \right) - \frac{4c_1^2}{9} - \frac{c_0^2}{3} - \frac{7c_1 c_0}{9} &= 0 \end{aligned} \quad (20)$$

Now using equations (19) and (20) we have

$$\begin{aligned} \frac{13c_1^2}{9} + \frac{25c_1 c_0}{9} + \frac{4c_0^2}{3} - \frac{4c_1^2}{9} - \frac{c_0^2}{3} - \frac{7c_1 c_0}{9} &= 3 \\ \Rightarrow c_1^2 + 2c_1 c_0 + c_0^2 &= 3 \Rightarrow c_1 + c_0 = \sqrt{3}. \end{aligned}$$

We already knew this information from equation (18), but we did not use it here. This means that two of our constraints are in fact only one. Hence we have one constraint less than expected, meaning that we are able to find a solution of our system since now the number of equations is the same as the number of unknowns. Recall that there are two underlying equations, which comes from the continuity condition of ω , which are

$$\frac{a}{3} = \frac{b_1}{3} + b_0 \quad \text{and} \quad \frac{2b_1}{3} + b_0 = \frac{2c_1}{3} + c_0$$

Let us continue to focus on the equation (19). Substitute $a = b_1 + 3b_0$, $b_0 = \frac{2c_1}{3} + c_0 - \frac{2b_1}{3}$ and $c_0 = \sqrt{3} - c_1$.

In other words, take

$$a = b_1 + 2c_1 + 3c_0 - 2b_1 = 2c_1 + 3c_0 - b_1 = 2c_1 + 3\sqrt{3} - 3c_1 - b_1 = 3\sqrt{3} - c_1 - b_1$$

and

$$b_0 = \frac{2c_1}{3} + \sqrt{3} - c_1 - \frac{2b_1}{3} = \sqrt{3} - \frac{c_1}{3} - \frac{2b_1}{3}$$

Hence we have that equation (19) becomes

$$\begin{aligned} -\sqrt{3} \left((3\sqrt{3} - c_1 - b_1) \frac{1}{18} + b_1 \frac{1}{6} + \left(\sqrt{3} - \frac{c_1}{3} - \frac{2b_1}{3} \right) \frac{1}{3} \right) + \frac{13c_1^2}{18} + \frac{25c_1(\sqrt{3} - c_1)}{18} + \frac{2(\sqrt{3} - c_1)^2}{3} &= \frac{3}{2} \\ \Rightarrow -\sqrt{3} \left(\frac{\sqrt{3}}{2} - \frac{c_1}{6} - \frac{b_1}{9} \right) + 2 + \frac{c_1 \sqrt{3}}{18} &= \frac{3}{2} \end{aligned}$$

$$\begin{aligned}\Rightarrow \frac{b_1\sqrt{3}}{9} + \frac{2c_1\sqrt{3}}{9} &= 1 \Rightarrow b_1 + 2c_1 = 3\sqrt{3}. \\ \Rightarrow b_1 &= 3\sqrt{3} - 2c_1.\end{aligned}$$

We can now consider the third equation of our system (i.e. equation (16)) and rewrite it in terms of c_1 . First, let us rewrite all the unknowns that we need in terms of c_1 :

$$a = 3\sqrt{3} - c_1 - b_1 = 3\sqrt{3} - c_1 - 3\sqrt{3} + 2c_1 = c_1$$

and

$$b_0 = \sqrt{3} - \frac{c_1}{3} - \frac{2b_1}{3} = \sqrt{3} - \frac{c_1}{3} - 2\sqrt{3} + \frac{4c_1}{3} = c_1 - \sqrt{3}$$

Notice that $a = c_1$ and $b_0 = -c_0$, which is in accordance with the results of Lyons and Victoir. We can now proceed with the substitution

$$\begin{aligned}& \frac{c_1^2}{81} + \frac{1}{3} \left(\frac{7}{27} (3\sqrt{3} - 2c_1)^2 + (3\sqrt{3} - 2c_1)(c_1 - \sqrt{3}) + (c_1 - \sqrt{3})^2 \right) \\ & + \frac{1}{3} \left(\frac{19}{27} c_1^2 + \frac{5}{3} c_1 (\sqrt{3} - c_1) + (\sqrt{3} - c_1)^2 \right) = \frac{1}{2\lambda_1(2H+1)} \\ \Rightarrow & \frac{c_1^2}{81} + \frac{1}{3} \left(7 + \frac{28c_1^2}{27} - \frac{28c_1\sqrt{3}}{9} + 3c_1\sqrt{3} - 9 - 2c_1^2 + 2c_1\sqrt{3} + c_1^2 + 3 - 2c_1\sqrt{3} \right) \\ & + \frac{1}{3} \left(\frac{19c_1^2}{27} + \frac{5c_1\sqrt{3}}{3} - \frac{5c_1^2}{3} + 3 + c_1^2 - 2c_1\sqrt{3} \right) = \frac{1}{2\lambda_1(2H+1)} \\ & c_1^2 \frac{1}{27} - c_1 \frac{4\sqrt{3}}{27} + \frac{4}{3} = \frac{3}{2H+1} \Rightarrow c_1^2 \frac{1}{27} - c_1 \frac{4\sqrt{3}}{27} = \frac{5-8H}{3(2H+1)} \\ \Rightarrow & c_1 = \frac{4H\sqrt{3} + 2\sqrt{3} - \sqrt{-96H^2 + 66H + 57}}{2H+1}\end{aligned}$$

If $H = \frac{1}{2}$ then

$$c_1 = \sqrt{3} \left(2 - \sqrt{\frac{11}{2}} \right)$$

which is in accordance with the results of Lyons and Victoir.

Therefore, we know all the unknowns. In particular, the path $\omega_{1,t}$ is given by:

$$\begin{cases} \frac{4H\sqrt{3} + 2\sqrt{3} - \sqrt{-96H^2 + 66H + 57}}{2H+1}t, & t \in [0, \frac{1}{3}], \\ \frac{2H\sqrt{3} + \sqrt{3} - \sqrt{-96H^2 + 66H + 57}}{2H+1} + \frac{2\sqrt{-96H^2 + 66H + 57} - 2H\sqrt{3} - \sqrt{3}}{2H+1}t, & t \in [\frac{1}{3}, \frac{2}{3}], \\ \frac{\sqrt{-96H^2 + 66H + 57} - 2H\sqrt{3} - \sqrt{3}}{2H+1} + \frac{4H\sqrt{3} + 2\sqrt{3} - \sqrt{-96H^2 + 66H + 57}}{2H+1}t, & t \in [\frac{2}{3}, 1], \end{cases}$$

If you let α and β to be:

$$\alpha := \frac{2H\sqrt{3} + \sqrt{3}}{2H + 1} \quad \text{and} \quad \beta := \frac{\sqrt{-96H^2 + 66H + 57}}{2H + 1}$$

then we can rewrite our path $\omega_{1,t}$ can be written in the following form:

$$\begin{cases} (2\alpha - \beta)t, & t \in [0, \frac{1}{3}], \\ (\alpha - \beta) + (2\beta - \alpha)t, & t \in [\frac{1}{3}, \frac{2}{3}], \\ (\beta - \alpha) + (2\alpha - \beta)t, & t \in [\frac{2}{3}, 1], \end{cases}$$

Now we proceed with a check of our result. Indeed we substitute the values obtained for our unknowns in our system of equations to check the correctness of these values.

In particular, let us focus first on the equation

$$\frac{a^2}{81} - 2(c_1 + c_0) \left(a\frac{1}{18} + b_1\frac{1}{6} + b_0\frac{1}{3} \right) + \frac{1}{3} \left(\frac{7}{27}b_1^2 + b_1b_0 + b_0^2 \right) + \frac{55c_1^2}{81} + \frac{2c_0^2}{3} + \frac{4c_1c_0}{3} = \frac{1}{2\lambda_1(2H+1)}$$

Let us rewrite it in terms of c_1 :

$$\begin{aligned} & \frac{c_1^2}{81} - 2\sqrt{3} \left(c_1\frac{1}{18} + (3\sqrt{3} - 2c_1)\frac{1}{6} + (c_1 - \sqrt{3})\frac{1}{3} \right) + \\ & + \frac{1}{3} \left(\frac{7}{27}(3\sqrt{3} - 2c_1)^2 + (3\sqrt{3} - 2c_1)(c_1 - \sqrt{3}) + (c_1 - \sqrt{3})^2 \right) \\ & + \frac{55c_1^2}{81} + \frac{2(\sqrt{3} - c_1)^2}{3} + \frac{4c_1(\sqrt{3} - c_1)}{3} = \frac{3}{2H+1} \\ & \Rightarrow c_1^2\frac{1}{27} - c_1\frac{4\sqrt{3}}{27} + \frac{4}{3} = \frac{3}{2H+1} \end{aligned}$$

as before. Now let us focus on the equation

$$(c_1 + c_0) \left(a\frac{1}{18} + b_1\frac{1}{6} + b_0\frac{1}{3} \right) - \frac{a^2}{81} - \frac{1}{3} \left(\frac{7}{27}b_1^2 + b_1b_0 + b_0^2 \right) + \frac{7c_1^2}{162} + \frac{c_1c_0}{18} = \frac{2H-1}{4\lambda_1(2H+1)}$$

Following the same procedure we have

$$\begin{aligned} & \sqrt{3} \left(c_1\frac{1}{18} + (3\sqrt{3} - 2c_1)\frac{1}{6} + (c_1 - \sqrt{3})\frac{1}{3} \right) - \frac{c_1^2}{81} \\ & - \frac{1}{3} \left(\frac{7}{27}(3\sqrt{3} - 2c_1)^2 + (3\sqrt{3} - 2c_1)(c_1 - \sqrt{3}) + (c_1 - \sqrt{3})^2 \right) \\ & + \frac{7c_1^2}{162} + \frac{c_1(\sqrt{3} - c_1)}{18} = \frac{3(2H-1)}{2(2H+1)} \\ & \Rightarrow -c_1^2\frac{1}{27} + c_1\frac{4\sqrt{3}}{27} + \frac{1}{6} = \frac{3(2H-1)}{2(2H+1)} \Rightarrow c_1^2\frac{1}{27} - c_1\frac{4\sqrt{3}}{27} = \frac{5-8H}{3(2H+1)}. \end{aligned}$$

as before. The other equations are straightforward to check.

Therefore, our solution is consistent and, as we have seen before, when $H = \frac{1}{2}$ our solution is the same solution obtained by Lyons and Victoir in their paper. \square

References

- [1] F. Baudoin and L. Coutin. Operators associated with a stochastic differential equation driven by fractional Brownian motions. *Stochastic Process. Appl.* 117 (2007), no. 5, 550–574. MR2320949
- [2] K. Chen. Integration of paths, geometric invariants and a generalized Baker-Hausdorff formula, *Annals of Mathematics*, Vol.65, No.1 (1957), pp.163-178.
- [3] K. Chen. Integration of paths – a faithful representation of paths by non-commutative formal power series. *Trans. Am. Math. Soc.*, 89:395-407, 1958.
- [4] K. Chen. Iterated path integrals, *Bulletin of the Am. Math. Soc.*, Vol.83, No.5 (1977), pp.831-879.
- [5] I. Chevyrev and T. Lyons. Characteristic functions of measures on geometric rough paths. To appear in the *Annals of Probability* arXiv: arXiv:1307.3580.
- [6] L. Coutin and Z. Qian. Stochastic analysis, rough path analysis and fractional Brownian motions. *Probab. Theory Relat. Field.*, 122(1):108140, 2002.
- [7] P. Friz and M. Hairer. A Course on Rough Paths, with an introduction to regularity structures. Springer (2014).
- [8] P. Friz and S. Riedel. Convergence rates for the full Gaussian rough paths. *Ann. Inst. H. Poincaré Probab. Statist.* Volume 50, Number 1 (2014), 154-194.
- [9] P. Friz, S. Riedel. Integrability of (non-)linear rough differential equations and integrals. *Stochastic Analysis and Applications*, Volume 31 (2013), Issue 2, Pages 336-358.
- [10] M. Hairer. A theory of regularity structures. *Inventiones mathematicae*, Volume 198, Issue 2, pp 269504 (2014).
- [11] T. J. Lyons. Differential equations driven by rough signals. *Rev. Mat. Iberoamericana* 14, no. 2, (1998), 215310. doi:10.4171/RMI/240.
- [12] T. J. Lyons, M. Caruana, and T. Lévy. Differential equations driven by rough paths, vol. 1908 of *Lecture Notes in Mathematics*. Springer, Berlin, (2007).
- [13] T. Lyons and N. Victoir. Cubature on Wiener space. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* 460, no. 2041, (2004), 169198. doi:10.1098/rspa.2003.1239. Stochastic analysis with applications to mathematical finance.
- [14] T. Lyons and N. Victoir. An extension theorem to rough paths. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 24, no. 5, (2007), 835847.
- [15] H. Ni, W. Xu, Concentration and exact convergence rates for expected Brownian signatures, *Electronic Communications in Probability*, Vol. 20, article 8, (2015).
- [16] L. C. Young. An inequality of the Hölder type, connected with Stieltjes integration. *Acta Math.* 67, no. 1, (1936), 251282.